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Online Appendix for "Persuasion via Weak Institutions"

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A Constructing an S-optimal Equilibrium

In this appendix, we informally explain how to use a (β, γ, k) that solves the program $(*)$ to construct a χ-equilibrium yielding S a value of $v^*_\chi(\mu_0)$. As a first step, let $(\sigma_\gamma, \alpha_\gamma, \pi_\gamma)$ denote an equilibrium of the cheap-talk game with modified prior γ that generates S payoff $\bar{v}(\gamma)$; some such equilibrium exists as we outlined in discussing the no-credibility case. If $k = 0$ (implying $\gamma = \mu_0$ by (χC)), then $(\xi, \sigma, \alpha, \pi) = (\sigma_\gamma, \sigma_\gamma, \alpha_\gamma, \pi_\gamma)$ is a χ-equilibrium delivering the desired S payoff.

Given the above observation, we can focus on the case in which every solution (β, γ, k) to the program has $k > 0$—or, equivalently, that $v^*_\chi(\mu_0) > \bar{v}(\mu_0)$. Let $\mathbf{B} \in \text{BP}(\beta, V_{\wedge \bar{v}(\gamma)})$ be such that $\int (\mu, s) \, \mathrm{d}\mathbf{B}(\mu, s) = (\beta, \hat{v}_{\wedge\gamma}(\beta))$. Lemma 3 in Appendix B.1.2 uses the geometry of concavification and quasiconcavification to prove $\mathbf{B}$ is supported only on outcomes in V that are left untouched by moving from V to $V_{\wedge \bar{v}(\gamma)}$. It follows $\mathbf{B}$ is in $\text{BP}(\beta, V)$, and so one can use the results from the full-credibility case to obtain some triple $(\xi_\beta, \alpha_\beta, \pi_\beta)$ that—when the prior is β and credibility level is $\chi = 1$—induces the outcome distribution $\mathbf{B}$, is consistent with Bayesian updating, and satisfies R's incentive constraints. Moreover, because the message space is rich, we may assume without loss that the messages M_β used by ξ_β have no overlap with the messages M_γ used by σ_γ.

Now, let us describe how the the above objects can be "pasted" together to deliver a χ-equilibrium with the relevant S payoff. Because constraint (BS) is satisfied, a binary signal can be used to "split" the prior into beliefs γ and β: concretely, some $\lambda : \Theta \to [0, 1]$ exists such that if message "high" and "low" are respectively sent

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with probability $1-\lambda(\theta)$ and $\lambda(\theta)$ in state θ, the would-be posterior distribution from hearing message "high" is γ and from "low" is β. Further, constraint (χC) implies $\lambda(\theta) \leq \chi$ for every state θ. We can therefore construct a χ-equilibrium as follows. The influencing S strategy σ is σ_γ; the official reporting protocol is given by

$$\xi(\theta) \coloneqq \xi^*(\theta)\xi_\beta(\theta) + [1-\xi^*(\theta)]\sigma_\gamma(\theta)$$

, where

$$\xi^*(\theta) \coloneqq 1 - \lambda(\theta)/\chi \in [0,1];$$

and the R strategy α and belief map π agree with $(\alpha_\beta, \pi_\beta)$ for messages in M_β and $(\alpha_\gamma, \pi_\gamma)$ for messages in M_γ. In the appendix, we show $(\xi, \sigma, \alpha, \pi)$ inherits the Bayesian and R incentive properties from its constituent pieces and generates an S payoff of $v^*_\chi(\mu_0)$. Moreover, because S is indifferent between all messages in M_γ and receives payoffs from $V_{\wedge \bar{v}(\gamma)}$ (hence, below $\bar{v}(\gamma)$) from messages in M_β, S's incentive constraints are also satisfied. Hence, we have found a χ-equilibrium generating S payoff $v^*_\chi(\mu_0)$, delivering the theorem.

A byproduct of the theorem's construction is the following result, which bounds the number of on-path messages required for an S-optimal equilibrium.

Corollary 1. *Some S-optimal χ-equilibrium exists with no more than* $\min\{|A|, 2|\Theta|-1\}$ *distinct messages sent on path.*

Existing literature has already established the above bounds hold when credibility is extreme. Specifically, Kamenica and Gentzkow (2011) and Lipnowski and Ravid (2020) note that when $\chi \in \{0,1\}$, an S-optimal χ-equilibrium exists that uses only $\min\{|A|, |\Theta|\}$ messages. Applying these bounds separately to **G** and **B** delivers that no χ-equilibrium S-value requires more than twice as many messages, that is, $\min\{2|A|, 2|\Theta|\}$. The corollary shows one can tighten these bounds by utilizing Theorem 1's construction. See Appendix B.1.3 for more details.

B Main Results

B.1 Toward the Proof of Theorem 1

Throughout this subsection, we work with a more general setting of the model in which both Θ and A are compact metrizable spaces with at least two elements, and the ob-

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jectives u_R and u_S are continuous.[16] Finally, we assume M is an uncountable compact metrizable space.[17] To generalize the definition of a χ-equilibrium and the value correspondence V, the sums are replaced with the corresponding integrals with respect to measures $\pi(m)$, $\alpha(m)$, and μ over Θ, A, and Θ, respectively. Further, throughout the appendix, we modify the definitions of the value function's concavification $\hat{v}$ (resp. quasiconcavification $\bar{v}$), letting it be the lowest (quasi)concave *and upper semicontinuous* function that dominates v.[18]

In addition, we allow for the possibility that credibility is state dependent, given by some measurable function $\boldsymbol{\chi} : \Theta \to [0,1]$. Throughout this appendix, we adopt the following notational convention. For a compact metrizable space Y, a probability measure $\mu \in \Delta Y$, and a function $f \colon Y \to \mathbb{R}$ that is bounded and measurable, let $f(\mu) := \int_Y f \,\mathrm{d}\mu \in \mathbb{R}$ denote the average value of f. In particular, for any credibility function $\boldsymbol{\chi}$, the scalar $\boldsymbol{\chi}(\mu_0)$ is simply the total probability that the report is not subject to influence.

Although accommodating this more general model entails some notational cost, all conceptual content of the proof is identical in the special case of constant credibility, and so the generalization requires no additional arguments. We therefore encourage the reader to read the entire proof while keeping in mind with the special case in which the function $\boldsymbol{\chi}$ is a constant χ.

We now provide a brief overview of the proof. Formalizing a form of *equilibrium summary* that is sufficient to calculate players' payoffs, the proof begins by showing an equivalence between the set of $\boldsymbol{\chi}$-equilibrium summaries, the set of $\boldsymbol{\chi}$-nonical equilibrium summaries, and the existence of a particular decomposition of the equilibrium distribution of R beliefs. This decomposition makes it easy to see program (2) is a relaxation of the program that maximizes S's value across all $\boldsymbol{\chi}$-equilibrium summaries. In particular, program (2) enables S to induce posteriors that would generate too high a continuation payoff for S. The proof's next part establishes this constraint is nonbinding at the optimum. We then conclude by explicitly writing the program that finds S's favorite equilibrium summary and showing its value is identical to that of (2).

[16] We view any compact metrizable space Y as a measurable space with its Borel field; let ΔY denote the set of all probability measures on Y; and endow ΔY with its weak* topology, so that ΔY is itself a compact metrizable space.

[17] In the special case in which A and $|\Theta|$ are finite, our characterization of sender-optimal equilibrium values (Theorem 1) and the propositions of section 4 hold if $|M| \geq \min\{|A|, 2|\Theta| - 1\}$; see Corollary 1.

[18] When Θ is finite, it follows from Carathéodory's theorem that the lowest (quasi)concave majorant of v is upper semicontinuous because v is. Hence, the present definition generalizes the one in the main text.

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B.1.1 Characterization of All Equilibrium Summaries

In this section, we characterize the full range of $\boldsymbol{\chi}$-equilibrium summaries, which we define below. In short, a $\boldsymbol{\chi}$-equilibrium summary consists of a description of the information R receives in equilibrium (which is jointly constructed by the official reporting protocol and an influencing S's messaging strategy), an expected payoff that S receives conditional on the official reporting protocol being used, and an expected payoff that S receives conditional on having the opportunity to influence.

To present unified proofs including for the case of $\boldsymbol{\chi} = \mathbf{1}$ and $\boldsymbol{\chi} = \mathbf{0}$, we adopt the notational convention that $\frac{0}{0} = 1$ wherever it appears.

We now define a convenient class of equilibria.

Definition 1. *A* ***$\boldsymbol{\chi}$-nonical equilibrium*** *is a $\boldsymbol{\chi}$-equilibrium $(\xi, \sigma, \alpha, \pi)$ such that every Borel $\hat{M} \subseteq M^*_\alpha$ has $\xi(\hat{M}|\cdot) = \xi(M^*_\alpha|\cdot)\ \sigma(\hat{M}|\cdot)$, where $M^*_\alpha := \operatorname{argmax}_{m\in M} u_S(\alpha(m))$.*

The above definition imposes further structure on a $\boldsymbol{\chi}$-equilibrium. The requirement pertains to the set M^*_α of the highest-payoff messages for S, which are necessarily the only messages an influencing S chooses. The condition says the conditional distribution of messages in M^*_α is identical for the official experiment and for an influencing sender's choices, in any state for which the official report sometimes sends messages in M^*_α. Informally, the condition says all differences in how the official and influenced report communicate are through whether they send a message in M^*_α in a given state.

Definition 2. *Say $(p, s_o, s_i) \in \Delta\Delta\Theta \times \mathbb{R} \times \mathbb{R}$ is a* ***$\boldsymbol{\chi}$-equilibrium summary*** *if some $\boldsymbol{\chi}$-equilibrium $(\xi, \sigma, \alpha, \pi)$ exists whose induced receiver belief distribution, official-report sender payoff, and influenced-report sender payoff are (p, s_o, s_i); that is,*

$$p = \left(\int_\Theta \left[\boldsymbol{\chi}\xi + (\mathbf{1} - \boldsymbol{\chi})\sigma\right] \mathrm{d}\mu_0\right) \circ \pi^{-1}$$

$$s_o = \int_\Theta \frac{\boldsymbol{\chi}}{\boldsymbol{\chi}(\mu_0)} \int_M u_S(\alpha(m))\, \mathrm{d}\xi(m|\cdot)\, \mathrm{d}\mu_0$$

$$s_i = \int_\Theta \frac{\mathbf{1}-\boldsymbol{\chi}}{1-\boldsymbol{\chi}(\mu_0)} \int_M u_S(\alpha(m))\, \mathrm{d}\sigma(m|\cdot)\, \mathrm{d}\mu_0.$$

If, further, $(\xi, \sigma, \alpha, \pi)$ is a $\boldsymbol{\chi}$-nonical equilibrium, we say (p, s_o, s_i) is a ***$\boldsymbol{\chi}$-nonical equilibrium summary****.*

Observe that knowing a $\boldsymbol{\chi}$-equilibrium's summary is sufficient for recovering each player's expected payoff: given a summary (p, s_o, s_i), S earns a payoff of $\boldsymbol{\chi}(\mu_0)s_o + [1 - \boldsymbol{\chi}(\mu_0)]\, s_i$, whereas R's expected utility is $\int_{\Delta\Theta} \max_{a\in A} \int_\Theta u_R(a, \cdot)\, \mathrm{d}\mu\, \mathrm{d}p(\mu)$.

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The following lemma adopts a belief-based approach, directly characterizing the range of $\boldsymbol{\chi}$-equilibrium summaries in our game. To state the characterization, let $\mathcal{P}(\mu) := \{p \in \Delta\Delta\Theta\colon\ \int \tilde{\mu}\,\mathrm{d}p(\tilde{\mu}) = \mu\}$ denote the set of **information policies** corresponding to prior $\mu \in \Delta\Theta$.

Lemma 1. *For $(p, s_o, s_i) \in \Delta\Delta\Theta \times \mathbb{R} \times \mathbb{R}$, the following are equivalent:*

1. *(p, s_o, s_i) is a $\boldsymbol{\chi}$-equilibrium summary;*
2. *(p, s_o, s_i) is a $\boldsymbol{\chi}$-nonical equilibrium summary;*
3. *Some $k \in [0,1]$, g, $b \in \Delta\Delta\Theta$ exist such that*
   - *(i)* $kb + (1-k)g = p \in \mathcal{P}(\mu_0)$;
   - *(ii)* $(1-k)\int_{\Delta\Theta} \mu\,\mathrm{d}g(\mu) \geq (\mathbf{1}-\boldsymbol{\chi})\mu_0$;
   - *(iii)* $g\{\mu \in \Delta\Theta : s_i \in V(\mu)\} = b\{\mu \in \Delta\Theta : \min V(\mu) \leq s_i\} = 1$;
   - *(iv)* $s_i - s_o \in \frac{k}{\boldsymbol{\chi}(\mu_0)}\left[s_i - \int_{\mathrm{supp}(b)} s_i \wedge V\,\mathrm{d}b\right]$.[19]

The first two parts of the lemma are self-explanatory. The third part says that the information policy p can be decomposed into two separate random posteriors, b and g, satisfying three conditions. Condition (ii) says the barycenter of g satisfies (χC). Condition (iii) says R is willing to give S a continuation payoff equal to s_i after all posteriors induced by g, and a lower continuation payoff for any posterior induced by b. And condition (iv) says R's best response to posteriors in b can be selected so that no posterior generates a payoff above s_i and so that S's average payoff conditional on her report coming from the official protocol adds up to s_o.

We now give an overview of Lemma 1. Obviously, 2 implies 1. Therefore, the proof proceeds by completing a cycle, showing 1 implies 3 and 3 implies 2. To show 1 implies 3, we take an equilibrium and partition the set of on-path messages into two subsets: the set of "good" messages for S to send (i.e., those that give S the highest possible expected payoff out of any possible message), and the complementary "bad"

[19] Here, $s_i \wedge V\colon \Delta\Theta \rightrightarrows \mathbb{R}$ is the correspondence with $s_i \wedge V(\mu) = (-\infty, s_i] \cap V(\mu)$; it is a Kakutani correspondence (because V is) on the restricted domain $\{\min V \leq s_i\} \supseteq \mathrm{supp}(b)$. The integral is the (Aumann) integral of a correspondence:

$$\int_{\mathrm{supp}(b)} s_i \wedge V\,\mathrm{d}b = \left\{\int_{\mathrm{supp}(b)} \phi\,\mathrm{d}b\colon \phi \text{ is a measurable selector of } s_i \wedge V|_{\mathrm{supp}(b)}\right\}.$$

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messages. Following this decomposition, one can obtain g and b by looking at the distribution of R's posterior beliefs conditional on the message being in the "good" or "bad" set, respectively. Letting k be the probability S sends a "bad" message, one obtains condition (i) from the usual Bayesian reasoning. Condition (ii) then follows from similar reasoning as explained in the main text, whereas conditions (iii) and (iv) follow from S's incentive constraints. To prove 3 implies 2, we use the decomposition provided by 3 to construct a $\boldsymbol{\chi}$-nonical equilibrium.

Proof. We show 1 implies 3 and 3 implies 2, noting 2 obviously implies 1.

Let us first show 1 implies 3. To that end, suppose $(\xi, \sigma, \alpha, \pi)$ is a $\boldsymbol{\chi}$-equilibrium resulting in summary (p, s_o, s_i). Let

$$G := \int_\Theta \sigma \, \mathrm{d}\left[\tfrac{\mathbf{1}-\boldsymbol{\chi}}{1-\boldsymbol{\chi}(\mu_0)}\mu_0\right] \text{ and } P := \int_\Theta [\boldsymbol{\chi}\xi + (\mathbf{1}-\boldsymbol{\chi})\sigma] \, \mathrm{d}\mu_0 \in \Delta M$$

denote the probability measures over messages induced by non-committed behavior and by average sender behavior, respectively. Let $k := 1 - P(M_\alpha^*)$ denote the ex-ante probability that a suboptimal message is sent. Sender incentive compatibility (which implies $\sigma(M_\alpha^*|\cdot) = \mathbf{1}$) tells us that $k \in [0, \boldsymbol{\chi}(\mu_0)]$. Let $B := \frac{1}{k}[P - (1-k)G]$ if $k > 0$; and let $B := \int_\Theta \xi \, \mathrm{d}\mu_0$ otherwise. As barycenters of probability measures over M, the measures G, P are in ΔM. Measure B on M therefore has total measure 1. Therefore, $B \in \Delta M$ as long as B is a positive measure, that is, $P \geq (1-k)G$. To see this measure inequality, note

$$(1-k)G = P(M_\alpha^*) \int_\Theta \sigma \, \mathrm{d}\left[\tfrac{\mathbf{1}-\boldsymbol{\chi}}{1-\boldsymbol{\chi}(\mu_0)}\mu_0\right] \leq \int_\Theta \sigma \, \mathrm{d}\left[(\mathbf{1}-\boldsymbol{\chi})\mu_0\right] \leq P,$$

where the first inequality follows from sender incentives (implying influenced reporting only sends messages in M_α^*). Now, define the induced belief distributions by these two distributions over messages, $g := G \circ \pi^{-1}$ and $b := B \circ \pi^{-1}$. By construction, $kb + (1-k)g = P \circ \pi^{-1} = p \in \mathcal{P}(\mu_0)$; that is, the first condition holds. Moreover, the second condition holds:

$$(1-k) \int_{\Delta\Theta} \mu \, \mathrm{d}g(\mu) = \int_M \pi \, \mathrm{d}[(1-k)G] = \int_{M_\alpha^*} \pi \, \mathrm{d}P \geq (\mathbf{1}-\boldsymbol{\chi})\mu_0,$$

where the inequality follows from the Bayesian property of π, together with the fact that σ almost surely sends a message from M_α^* on the path of play. Next, observe

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that for any $m \in M$, sender incentive compatibility tells us $u_S(\alpha(m)) \leq s_i$, and receiver incentive compatibility implies $\alpha(m) \in V(\pi(m))$. It follows directly that $g\{V \ni s_i\} = b\{\min V \leq s_i\} = 1$; that is, the third condition holds. Toward the fourth and final condition, let us view π, α as random variables on the probability space $\langle M, P\rangle$. Defining the conditional expectation $\phi_0 := \mathbb{E}_B[u_S(\alpha)|\pi] : M \to \mathbb{R}$, the Doob-Dynkin lemma delivers a measurable function $\phi : \Delta\Theta \to \mathbb{R}$ such that $\phi \circ \pi =_{B-\text{a.e.}} \phi_0$. Because $u_S(\alpha(m)) \in s_i \wedge V(m)$ for every $m \in M$, and the correspondence $s_i \wedge V$ is compact- and convex-valued, it must be that $\phi_0 \in_{B-\text{a.e.}} s_i \wedge V(\pi)$. Therefore, $\phi \in_{b-\text{a.e.}} s_i \wedge V$. Modifying ϕ on a b-null set, we may assume without loss that ϕ is a measurable selector of $s_i \wedge V$. Observe now that

$$\int_{\text{supp}(b)} \phi \,\mathrm{d}b = \int_M \phi_0 \,\mathrm{d}B = \int_M \mathbb{E}_B[u_S(\alpha)|\pi] \,\mathrm{d}B = \int_M u_S \circ \alpha \,\mathrm{d}B.$$

Therefore, because $G(M^*_\alpha) = 1$,

$$\begin{aligned} s_o &= \int_M u_S \circ \alpha \,\mathrm{d}\tfrac{P-[1-\boldsymbol{\chi}(\mu_0)]G}{\boldsymbol{\chi}(\mu_0)} = \int_M u_S \circ \alpha \,\mathrm{d}\tfrac{kB+(1-k)G-[1-\boldsymbol{\chi}(\mu_0)]G}{\boldsymbol{\chi}(\mu_0)} \\ &= \int_M u_S \circ \alpha \,\mathrm{d}\left[\left(1-\tfrac{k}{\boldsymbol{\chi}(\mu_0)}\right)G + \tfrac{k}{\boldsymbol{\chi}(\mu_0)}B\right] = \left(1-\tfrac{k}{\boldsymbol{\chi}(\mu_0)}\right)s_i + \tfrac{k}{\boldsymbol{\chi}(\mu_0)}\int_{\text{supp}(b)} \phi \,\mathrm{d}b, \end{aligned}$$

as required.

Now, we show 3 implies 2. Because M is an uncountable Polish space, the Borel isomorphism theorem (Theorem 3.3.13 Srivastava, 2008) says M is isomorphic (as a measurable space) to $\{i, o\} \times \Delta\Theta$. We can therefore assume without loss that $M = \{i, o\} \times \Delta\Theta$.

Suppose $k \in [0, 1]$, $g, b \in \Delta\Delta\Theta$ satisfy the four listed conditions so that 3 holds, and let ϕ be a measurable selector of $s_i \wedge V|_{\text{supp}(b)}$ with $s_o = \left(1-\frac{k}{\boldsymbol{\chi}(\mu_0)}\right)s_i + \frac{k}{\boldsymbol{\chi}(\mu_0)}\int_{\text{supp}(b)} \phi \,\mathrm{d}b$, which the fourth condition assures us exists.

We construct a $\boldsymbol{\chi}$-nonical equilibrium from these objects that induces summary (p, s_o, s_i).

Let us proceed in two cases. First, consider the case in which $s_o = s_i$. In this case, the fourth condition implies $b\{\phi = s_i\} = 1$, so that $p \in \mathcal{P}(\mu_0)$ has $p\{V \ni s_i\} = 1$. Hence, (V being upper hemicontinuous) Lipnowski and Ravid (2020, Lemma 1) delivers an equilibrium (σ, α, π) of the pure cheap-talk game generating receiver information distribution p and sender payoff s_i. It follows immediately that $(\sigma, \sigma, \alpha, \pi)$ is a $\boldsymbol{\chi}$-nonical equilibrium that induces summary (p, s_i, s_i).

Henceforth, we focus on the remaining case in which $s_o < s_i$. Without loss of generality, we may further assume $b\{\phi < s_i\} = 1$.[20] Define $\beta := \int_{\Delta\Theta} \mu \, \mathrm{d}b(\mu)$ and $\gamma := \int_{\Delta\Theta} \mu \, \mathrm{d}g(\mu)$. Let measurable $\eta_g : \Theta \to \Delta[\mathrm{supp}(g)] \subseteq \Delta\Delta\Theta$ and $\eta_b : \Theta \to \Delta[\mathrm{supp}(b)] \subseteq \Delta\Delta\Theta$ be signals that induce belief distribution g for prior γ and belief distribution b for prior β, respectively, such that for each such signal the induced posterior belief is to equal the message itself. That is, for every Borel $\hat{\Theta} \subseteq \Theta$ and $\hat{D} \subseteq \Delta\Theta$,

$$\int_{\hat{\Theta}} \eta_b(\hat{D}|\cdot) \, \mathrm{d}\beta = \int_{\hat{D}} \mu(\hat{\Theta}) \, \mathrm{d}b(\mu) \text{ and } \int_{\hat{\Theta}} \eta_g(\hat{D}|\cdot) \, \mathrm{d}\gamma = \int_{\hat{D}} \mu(\hat{\Theta}) \, \mathrm{d}g(\mu).$$

Take some Radon-Nikodym derivative $\frac{\mathrm{d}\beta}{\mathrm{d}\mu_0} : \Theta \to \mathbb{R}_+$; changing it on a μ_0-null set, we may assume $\mathbf{0} \leq \frac{k}{\boldsymbol{\chi}} \frac{\mathrm{d}\beta}{\mathrm{d}\mu_0} \leq \mathbf{1}$ because $(1-k)\gamma \geq (\mathbf{1} - \boldsymbol{\chi})\mu_0$. With the above ingredients in hand, we can define the sender's influenced strategy and reporting protocol

$$\begin{aligned} \sigma &:= \delta_i \otimes \eta_g : \Theta \to \Delta M, \\ \xi &:= \left(\mathbf{1} - \frac{k}{\boldsymbol{\chi}} \frac{\mathrm{d}\beta}{\mathrm{d}\mu_0}\right) \delta_i \otimes \eta_g + \frac{k}{\boldsymbol{\chi}} \frac{\mathrm{d}\beta}{\mathrm{d}\mu_0} \delta_o \otimes \eta_b : \Theta \to \Delta M. \end{aligned}$$

Because $M_i := \{i\} \times \Delta\Theta$ obviously has $\sigma(M_i|\cdot) = \mathbf{1}$ and $\xi(\hat{M}_i|\cdot) = \xi(M_i|\cdot)\ \sigma(\hat{M}_i|\cdot)$ for every Borel $\hat{M}_i \subseteq M_i$, it follows that a $\boldsymbol{\chi}$-equilibrium with sender play described by (σ, ξ) is in fact a $\boldsymbol{\chi}$-nonical equilibrium, as long as the receiver strategy α satisfies $M^*_\alpha \supseteq M_i$. To finish constructing such a $\boldsymbol{\chi}$-equilibrium, we define the receiver strategy and belief map for our proposed equilibrium as follows. Intuitively, an on-path message (i, μ) will lead to belief μ and a receiver best response that delivers payoff s_i to the sender; an on-path message (o, μ) will lead to belief μ and a receiver best response that delivers a potentially lower payoff to the sender, calibrated to give the target average payoff; and off-path messages are interpreted as equivalent to some on-path message so as not to introduce new incentive constraints. Formally, fix some $\hat{\mu} \in \mathrm{supp}(b)$, which will serve as a default belief and incentive-compatible receiver response for any off-path

[20] Indeed, one could replace k with $\tilde{k} := kb\{\phi < s_i\} > 0$, replace b with $\tilde{b} := \frac{k}{\tilde{k}} b\left((\cdot) \cap \{\phi < s_i\}\right)$, and replace g with $\tilde{g} := \frac{1}{1-\tilde{k}}(p - \tilde{k}\tilde{b})$.

messages. We can then define a receiver belief map as

$$\pi : M \to \Delta\Theta$$
$$m \mapsto \begin{cases} \mu & : \ m = (i,\mu) \text{ for } \mu \in \text{supp}(g), \ \text{ or } m = (o,\mu) \text{ for } \mu \in \text{supp}(b) \\ \hat{\mu} & : \ \text{otherwise.} \end{cases}$$

Finally, by Lipnowski and Ravid (2020, Lemma 2), some measurable $\alpha_b, \alpha_g : \Delta\Theta \to \Delta A$ exist such that[21]

- $\alpha_b(\mu), \alpha_g(\mu) \in \text{argmax}_{\tilde{\alpha} \in \Delta A}\, u_R(\tilde{\alpha}, \mu)\ \forall \mu \in \Delta\Theta$;
- $u_S(\alpha_b(\mu)) = \phi(\mu)\ \forall \mu \in \text{supp}(b)$, and $u_S(\alpha_g(\mu)) = s_i\ \forall \mu \in \text{supp}(g)$.

From these selectors, we can define a receiver strategy as

$$\alpha : M \to \Delta A$$
$$m \mapsto \begin{cases} \alpha_b(\mu) & : \ m = (o,\mu) \text{ for some } \mu \in \text{supp}(b) \\ \alpha_g(\mu) & : \ m = (i,\mu) \text{ for some } \mu \in \text{supp}(g) \\ \alpha_b(\hat{\mu}) & : \ \text{otherwise.} \end{cases}$$

We want to show the tuple $(\xi, \sigma, \alpha, \pi)$ is a $\boldsymbol{\chi}$-equilibrium (hence, a $\boldsymbol{\chi}$-nonical equilibrium) resulting in summary (p, s_o, s_i). It is immediate from the construction of (σ, α, π) that sender incentive compatibility and receiver incentive compatibility hold, and that the expected sender payoff is s_i given influenced reporting. It remains to verify that the induced receiver belief distribution is p, that the Bayesian property is satisfied, and that the expected sender payoff from the official report is s_o. We verify these features below, via a tedious computation.

Recall $\boldsymbol{\chi}\xi : \Theta \to \Delta M$ is defined as the pointwise product; that is, for every $\theta \in \Theta$ and Borel $\hat{M} \subseteq M$, we have $(\boldsymbol{\chi}\xi)(\hat{M}|\theta) = \boldsymbol{\chi}(\theta)\xi(\hat{M}|\theta)$; and similarly for $(\mathbf{1} - \boldsymbol{\chi})\sigma$. To

[21]The cited lemma delivers $\alpha_b|_{\text{supp}(b)}, \alpha_g|_{\text{supp}(g)}$. Then, as $\text{supp}(p) \subseteq \text{supp}(b) \cup \text{supp}(g)$, we can extend both functions to the rest of their domains by making them agree on $\text{supp}(p) \setminus [\text{supp}(b) \cap \text{supp}(g)]$.

see that the Bayesian property holds, observe that every Borel $D \subseteq \Delta\Theta$ satisfies

$$\begin{aligned}[][(\mathbf{1}-\boldsymbol{\chi})\sigma+\boldsymbol{\chi}\xi](\{o\}\times D|\cdot) &= k\tfrac{\mathrm{d}\beta}{\mathrm{d}\mu_0}\eta_b(D|\cdot)\\ [(\mathbf{1}-\boldsymbol{\chi})\sigma+\boldsymbol{\chi}\xi](\{i\}\times D|\cdot) &= \left[(\mathbf{1}-\boldsymbol{\chi})+\boldsymbol{\chi}\left(\mathbf{1}-\tfrac{k}{\boldsymbol{\chi}}\tfrac{\mathrm{d}\beta}{\mathrm{d}\mu_0}\right)\right]\eta_g(D|\cdot)\\ &= \left(\mathbf{1}-k\tfrac{\mathrm{d}\beta}{\mathrm{d}\mu_0}\right)\eta_g(D|\cdot).\end{aligned}$$

Now, take any Borel $\hat{M} \subseteq M$ and $\hat{\Theta} \subseteq \Theta$, and let $D_z := \left\{\mu \in \Delta\Theta :\ (z,\mu) \in \hat{M}\right\}$ for $z \in \{i, o\}$. Observe that

$$\begin{aligned}&\int_\Theta \int_{\hat{M}} \pi(\hat{\Theta}|m)\,\mathrm{d}[(1-\boldsymbol{\chi})\sigma+\boldsymbol{\chi}\xi](m|\cdot)\,\mathrm{d}\mu_0\\ &= \int_\Theta \left(\int_{\{o\}\times D_o}+\int_{\{i\}\times D_i}\right)\pi(\hat{\Theta}|m)\,\mathrm{d}[(1-\boldsymbol{\chi})\sigma+\boldsymbol{\chi}\xi](m|\cdot)\,\mathrm{d}\mu_0\\ &= \int_\Theta \left[k\tfrac{\mathrm{d}\beta}{\mathrm{d}\mu_0}\int_{D_o}\mu(\hat{\Theta})\,\mathrm{d}\eta_b(\mu|\cdot)+\left(1-k\tfrac{\mathrm{d}\beta}{\mathrm{d}\mu_0}\right)\int_{D_i}\mu(\hat{\Theta})\,\mathrm{d}\eta_g(\mu|\cdot)\right]\mathrm{d}\mu_0\\ &= k\int_\Theta\int_{D_o}\mu(\hat{\Theta})\,\mathrm{d}\eta_b(\mu|\cdot)\,\mathrm{d}\beta+\int_\Theta\int_{D_i}\mu(\hat{\Theta})\,\mathrm{d}\eta_g(\mu|\cdot)\,\mathrm{d}[\mu_0-k\beta]\\ &= k\int_\Theta\int_{D_o}\mu(\hat{\Theta})\,\mathrm{d}\eta_b(\mu|\cdot)\,\mathrm{d}\beta+(1-k)\int_\Theta\int_{D_i}\mu(\hat{\Theta})\,\mathrm{d}\eta_g(\mu|\cdot)\,\mathrm{d}\gamma\\ &= k\int_{D_o}\int_\Theta\mu(\hat{\Theta})\,\mathrm{d}\mu(\theta)\,\mathrm{d}b(\mu)+(1-k)\int_{D_i}\int_\Theta\mu(\hat{\Theta})\,\mathrm{d}\mu(\theta)\,\mathrm{d}g(\mu)\\ &= k\int_{D_o}\mu(\hat{\Theta})\,\mathrm{d}b(\mu)+(1-k)\int_{D_i}\mu(\hat{\Theta})\,\mathrm{d}g(\mu).\end{aligned}$$

Let us establish that the above computation implies both that (ξ, σ, π) satisfies the Bayesian property (making $(\xi, \sigma, \alpha, \pi)$ a $\boldsymbol{\chi}$-equilibrium) and that its induced belief distribution is p. First, observe that

$$\begin{aligned}&\int_\Theta \int_{\hat{M}} \pi(\hat{\Theta}|m)\,\mathrm{d}[(1-\boldsymbol{\chi})\sigma+\boldsymbol{\chi}\xi](m|\cdot)\,\mathrm{d}\mu_0\\ &= k\int_{D_o}\mu(\hat{\Theta})\,\mathrm{d}b(\mu)+(1-k)\int_{D_i}\mu(\hat{\Theta})\,\mathrm{d}g(\mu)\\ &= k\int_{\hat{\Theta}}\eta_b(D_o|\cdot)\,\mathrm{d}\beta+(1-k)\int_{\hat{\Theta}}\eta_g(D_i|\cdot)\,\mathrm{d}\gamma\\ &= \int_{\hat{\Theta}}\eta_b(D_o|\cdot)\,\mathrm{d}[k\beta]+\int_{\hat{\Theta}}\eta_g(D_i|\cdot)\,\mathrm{d}[\mu_0-k\beta]\\ &= \int_{\hat{\Theta}}\left[k\tfrac{\mathrm{d}\beta}{\mathrm{d}\mu_0}\eta_b(D_o|\cdot)+\left(\mathbf{1}-k\tfrac{\mathrm{d}\beta}{\mathrm{d}\mu_0}\right)\eta_g(D_i|\cdot)\right]\mathrm{d}\mu_0\end{aligned}$$

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$$= \int_{\hat{\Theta}} [(\mathbf{1}-\boldsymbol{\chi})\sigma + \boldsymbol{\chi}\xi](\hat{M}|\cdot)\, \mathrm{d}\mu_0,$$

verifying the Bayesian property. Second, for any Borel $D \subseteq \Delta\Theta$, we can specialize to the case of $D_o = D_i = D$ and $\hat{\Theta} = \Theta$, showing the equilibrium probability of the receiver posterior belief belonging to D is exactly

$$\int_{\Theta} [(\mathbf{1}-\boldsymbol{\chi})\sigma + \boldsymbol{\chi}\xi](\{i,o\} \times D|\cdot)\, \mathrm{d}\mu_0 = k \int_D \mathbf{1}\, \mathrm{d}b + (1-k)\int_D \mathbf{1}\, \mathrm{d}g = p(D).$$

Finally, the expected sender payoff conditional on reporting not being influenced is given by

$$\begin{aligned}
&\int_{\Theta}\int_M u_S(\alpha(m))\, \mathrm{d}\xi(m|\cdot)\, \mathrm{d}\left[\tfrac{\boldsymbol{\chi}}{\boldsymbol{\chi}(\mu_0)}\mu_0\right] \\
&= \int_{\Theta}\left[\left(1 - \tfrac{k}{\boldsymbol{\chi}}\tfrac{\mathrm{d}\beta}{\mathrm{d}\mu_0}\right)\int_{\Delta\Theta} u_S(\alpha(i,\mu))\, \mathrm{d}\eta_g(\mu|\cdot) + \tfrac{k}{\boldsymbol{\chi}}\tfrac{\mathrm{d}\beta}{\mathrm{d}\mu_0}\int_{\Delta\Theta} u_S(\alpha(o,\mu))\, \mathrm{d}\eta_b(\mu|\cdot)\right] \mathrm{d}\left[\tfrac{\boldsymbol{\chi}}{\boldsymbol{\chi}(\mu_0)}\mu_0\right] \\
&= \int_{\Theta}\left[\left(\mathbf{1} - \tfrac{k}{\boldsymbol{\chi}}\tfrac{\mathrm{d}\beta}{\mathrm{d}\mu_0}\right)\int_{\Delta\Theta} s_i\, \mathrm{d}\eta_g(\mu|\cdot) + \tfrac{k}{\boldsymbol{\chi}}\tfrac{\mathrm{d}\beta}{\mathrm{d}\mu_0}\int_{\mathrm{supp}(b)} \phi(\mu)\, \mathrm{d}\eta_b(\mu|\cdot)\right] \mathrm{d}\left[\tfrac{\boldsymbol{\chi}}{\boldsymbol{\chi}(\mu_0)}\mu_0\right] \\
&= s_i + \tfrac{k}{\boldsymbol{\chi}(\mu_0)}\int_{\Theta}\left[-s_i + \int_{\mathrm{supp}(b)} \phi(\mu)\, \mathrm{d}\eta_b(\mu|\theta)\right] \mathrm{d}\beta(\theta) \\
&= \left[1 - \tfrac{k}{\boldsymbol{\chi}(\mu_0)}\right] s_i + \tfrac{k}{\boldsymbol{\chi}(\mu_0)}\int_{\Delta\Theta}\int_{\Theta} \phi(\mu)\, \mathrm{d}\mu(\theta)\, \mathrm{d}b(\mu) \\
&= \tfrac{(1-k)-[1-\boldsymbol{\chi}(\mu_0)]}{\boldsymbol{\chi}(\mu_0)} s_i + \tfrac{k}{\boldsymbol{\chi}(\mu_0)}\int_{\mathrm{supp}(b)} \phi\, \mathrm{d}b \\
&= s_o,
\end{aligned}$$

as required.

□

B.1.2 Proof of Theorem 1

We begin with a simple technical lemma on the geometry of concavifications and the belief distributions that attain them.

Lemma 2. *If $f : \Delta\Theta \to \mathbb{R}$ is upper semicontinuous, $\hat{f}$ is f's concavification, $\beta \in \Delta\Theta$, and $b \in \mathcal{P}(\beta)$ has $\int f\, \mathrm{d}b = \hat{f}(\beta)$, then $b\{\mu \in \Delta\Theta : \ \hat{f}|_{\mathrm{co}\{\beta,\mu\}}$ affine$\} = 1$.*

Proof. First, observe that every concave, non-affine function $\varphi : [0,1] \to \mathbb{R}$ has $\varphi(z) > z\varphi(1) + (1-z)\varphi(0)$ for every $z \in (0,1)$. Hence, it suffices to show $\hat{f}\left(\tfrac{1}{2}\beta + \tfrac{1}{2}\mu\right) =$

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$\frac{1}{2}\hat{f}(\beta)+\frac{1}{2}\hat{f}(\mu)$ a.s.-$b(\mu)$. Equivalently, because concavity of $\hat{f}$ implies $\frac{1}{2}\hat{f}(\beta)+\frac{1}{2}\hat{f}(\mu)-\hat{f}\left(\frac{1}{2}\beta+\frac{1}{2}\mu\right)\leq 0$ for every $\mu\in\Delta\Theta$, we need only show $\int\left[\frac{1}{2}\hat{f}(\beta)+\frac{1}{2}\hat{f}(\mu)\right]\,\mathrm{d}b(\mu)$ and $\int f\left(\frac{1}{2}\beta+\frac{1}{2}\mu\right)\,\mathrm{d}b(\mu)$ coincide. To show this identity, observe that (because $\hat{f}$ is concave, upper semicontinuous, and everywhere above f)

$$\hat{f}(\beta)=\int\left[\tfrac{1}{2}\hat{f}(\beta)+\tfrac{1}{2}f\right]\,\mathrm{d}b\leq\int\left[\tfrac{1}{2}\hat{f}(\beta)+\tfrac{1}{2}\hat{f}\right]\,\mathrm{d}b$$
$$\leq\int\hat{f}\left(\tfrac{1}{2}\beta+\tfrac{1}{2}\mu\right)\,\mathrm{d}b(\mu)\leq\hat{f}\left(\int\left[\tfrac{1}{2}\beta+\tfrac{1}{2}\mu\right]\,\mathrm{d}b(\mu)\right)=\hat{f}(\beta).$$

Hence, all of the above expressions are equal, delivering the lemma. □

Before proceeding to the proof of Theorem 1, we prove a useful lemma about the theorem's auxiliary program. In short, the lemma shows that a relaxation built into this program—that S can be held to payoff $\bar{v}(\gamma)$ even at beliefs at which every R best response gives S a higher payoff—is payoff irrelevant at an optimum.

Lemma 3. *If (β,γ,k) solve program (2) and have $\hat{v}_{\wedge\gamma}(\beta)<\bar{v}(\gamma)$, and $b\in\mathcal{P}(\beta)$ has $\int v_{\wedge\gamma}\,\mathrm{d}b=\hat{v}_{\wedge\gamma}(\beta)$, then $v_{\wedge\gamma}(\mu)\in V(\mu)$ for every $\mu\in\operatorname{supp}(b)$. In particular, $b\{\min V\leq\bar{v}(\gamma)\}=1$.*

Proof. Given the definition of $v_{\wedge\gamma}$, and given that V is nonempty-compact-convex-valued, it suffices to show $w(\mu)\leq\bar{v}(\gamma)$ for $\mu\in\operatorname{supp}(b)$, where $w:=\min V$. Then, because V is upper hemicontinuous, it suffices to show $b\{w\leq\bar{v}(\gamma)\}=1$. To that end, define $D:=\left\{\mu\in\Delta\Theta:\ \hat{v}_{\wedge\gamma}|_{\mathrm{co}\{\beta,\mu\}}\text{ affine}\right\}$. Applying Lemma 2 to $v_{\wedge\gamma}$ implies $b(D)=1$, so the lemma will follow if we can show $w|_D\leq\bar{v}(\gamma)$.

Let us establish that every $\mu\in D$ has $w(\mu)\leq\bar{v}(\gamma)$. The result is obvious if $v(\mu)<\bar{v}(\gamma)$, so we focus on the case in which $v(\mu)\geq\bar{v}(\gamma)$. For such μ, note every proper convex combination μ' of β and μ has $v(\mu')<\bar{v}(\gamma)$; otherwise, $\hat{v}_{\wedge\gamma}(\beta)<\hat{v}_{\wedge\gamma}(\mu')=\hat{v}_{\wedge\gamma}(\mu)$, violating the definition of $D\ni\mu$. It follows that μ is in the closure of $\{v\leq\bar{v}(\gamma)\}\subseteq\{w\leq\bar{v}(\gamma)\}$. Lower semicontinuity of w then implies $w(\mu)\leq\bar{v}(\gamma)$. □

We now prove our main theorem: an S-optimal $\boldsymbol{\chi}$-equilibrium exists, giving S payoff $v^*_{\boldsymbol{\chi}}(\mu_0)$.

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Proof. By Lemma 1, the supremum sender value over all $\boldsymbol{\chi}$-equilibrium summaries is

$$\begin{aligned}\tilde{v}^*_{\boldsymbol{\chi}}(\mu_0) := \sup_{b,g\in\Delta\Delta\Theta,\ k\in[0,1],\ s_o,s_i\in\mathbb{R}} & \left\{\boldsymbol{\chi}(\mu_0)s_o + [1-\boldsymbol{\chi}(\mu_0)]s_i\right\}\\ \text{s.t.} \quad & kb+(1-k)g\in\mathcal{P}(\mu_0),\ (1-k)\int_{\Delta\Theta}\mu\,\mathrm{d}g(\mu)\geq(\mathbf{1}-\boldsymbol{\chi})\mu_0,\\ & g\{V\ni s_i\}=b\{\min V\leq s_i\}=1,\\ & s_o\in\left(1-\tfrac{k}{\boldsymbol{\chi}(\mu_0)}\right)s_i+\tfrac{k}{\boldsymbol{\chi}(\mu_0)}\int_{\text{supp}(b)}s_i\wedge V\,\mathrm{d}b.\end{aligned}$$

Given any feasible (b,g,k,s_o,s_i) in the above program, replacing the associated measurable selector of $s_i\wedge V|_{\text{supp}(b)}$ with the weakly higher function $s_i\wedge v|_{\text{supp}(b)}$, and raising s_o to $\left(1-\frac{k}{\boldsymbol{\chi}(\mu_0)}\right)s_i+\frac{k}{\boldsymbol{\chi}(\mu_0)}\int_{\text{supp}(b)}s_i\wedge v\,\mathrm{d}b$, weakly raises the objective and preserve all constraints. Therefore,

$$\begin{aligned}\tilde{v}^*_{\boldsymbol{\chi}}(\mu_0) = \sup_{b,g\in\Delta\Delta\Theta,\ k\in[0,1],\ s_i\in\mathbb{R}} & \left\{(1-k)s_i+k\int_{\text{supp}(b)}s_i\wedge v\,\mathrm{d}b\right\}\\ \text{s.t.} \quad & kb+(1-k)g\in\mathcal{P}(\mu_0),\ (1-k)\int_{\Delta\Theta}\mu\,\mathrm{d}g(\mu)\geq(\mathbf{1}-\boldsymbol{\chi})\mu_0,\\ & g\{V\ni s_i\}=b\{\min V\leq s_i\}=1.\end{aligned}$$

Given any feasible (b,g,k,s_i) in the latter program, replacing (g,s_i) with any (g^*,s_i^*) such that $\int_{\Delta\Theta}\mu\,\mathrm{d}g^*(\mu)=\int_{\Delta\Theta}\mu\,\mathrm{d}g(\mu)$, $g^*\{V\ni s_i^*\}=1$, and $s_i^*\geq s_i$ will preserve all constraints and weakly raise the objective. Moreover, Lipnowski and Ravid (2020, Lemma 1 and Theorem 2) tell us that any $\gamma\in\Delta\Theta$ has $\max_{g\in\mathcal{P}(\gamma),s_i\in\mathbb{R}:\ g\{V\ni s_i\}=1}s_i=\bar{v}(\gamma)$.[22] Therefore,

$$\begin{aligned}\tilde{v}^*_{\boldsymbol{\chi}}(\mu_0) = \sup_{\beta,\gamma\in\Delta\Theta,\ k\in[0,1],\ b\in\mathcal{P}(\beta)} & \left\{(1-k)\bar{v}(\gamma)+k\int_{\Delta\Theta}v_{\wedge\gamma}\,\mathrm{d}b\right\}\\ \text{s.t.} \quad & k\beta+(1-k)\gamma=\mu_0,\ (1-k)\gamma\geq(\mathbf{1}-\boldsymbol{\chi})\mu_0,\\ & b\{\min V\leq\bar{v}(\gamma)\}=1.\end{aligned}$$

Trivially, the program (2) that defines $v^*_{\boldsymbol{\chi}}(\mu_0)$ is a relaxation of the above program; that is, for every feasible (β,γ,k,b) for the above program, (β,γ,k) is feasible in (2) and generates a weakly higher objective there; that is, $\tilde{v}^*_{\boldsymbol{\chi}}(\mu_0)\leq v^*_{\boldsymbol{\chi}}(\mu_0)$. We now prove the opposite inequality also holds, thereby completing the theorem's proof. Notice the

[22] Note $g\{V\ni s_i\}=1$ implies $s_i\in\bigcap_{\mu\in\text{supp}(g)}V(\mu)$ because V is upper hemicontinuous.

program (2) has an upper-semicontinuous objective and compact constraint set, and so admits some solution (β, γ, k). We now argue some $(\tilde{\beta}, \tilde{\gamma}, \tilde{k}, b)$ exists that is feasible for the above program and such that

$$(1-\tilde{k})\bar{v}(\tilde{\gamma}) + \tilde{k}\int v_{\wedge\tilde{\gamma}}\,\mathrm{d}b \geq k\hat{v}_{\wedge\gamma}(\beta) + (1-k)\bar{v}(\gamma),$$

and so $\tilde{v}^*_{\boldsymbol{\chi}}(\mu_0) \geq v^*_{\boldsymbol{\chi}}(\mu_0)$. If $\hat{v}_{\wedge\gamma}(\beta) < \bar{v}(\gamma)$, Lemma 3 delivers b such that (β, γ, k, b) is as desired. Otherwise, $\hat{v}_{\wedge\gamma}(\beta) = \bar{v}(\gamma)$, and so quasiconcavity of $\bar{v}$ implies $\bar{v}(\mu_0) \geq k\hat{v}_{\wedge\gamma}(\beta) + (1-k)\bar{v}(\gamma)$, meaning $(\mu_0, \mu_0, 0, \delta_{\mu_0})$ is as desired. The theorem follows. □

B.1.3 Simple Communication: Proof of Corollary 1

We begin with a lemma showing program (2) always admits a solution with additional structure. In particular, whenever S-optimal $\boldsymbol{\chi}$-equilibrium requires the official reporting protocol to differ from an influencing S's behavior, we can assume without loss that every message sent by official reporting is *strictly* suboptimal for an influencing S.

Lemma 4. *One of the following holds:*

1. *The triple* $(\beta, \gamma, k) = (\mu_0, \mu_0, 0)$ *is an optimal solution to program* (2)*;*
2. *Some optimal solution* (β, γ, k) *to program* (2) *and* $b \in \mathcal{P}(\beta)$ *exist with* $k > 0$*,* $\int v_{\wedge\gamma}\,\mathrm{d}b = \hat{v}_{\wedge\gamma}(\beta)$*, and* $b\{v < \bar{v}(\gamma)\} = 1$.

Proof. As observed in (the SDC generalization of) Theorem 1, program (2) admits some solution (β, γ, k). Further, some $b \in \mathcal{P}(\beta)$ exists with $\int v_{\wedge\gamma}\,\mathrm{d}b = \hat{v}_{\wedge\gamma}(\beta)$ because $\mathcal{P}(\beta)$ is compact and $b \mapsto \int v_{\wedge\gamma}\,\mathrm{d}b$ is upper semicontinuous. Letting $D := \{v \geq \bar{v}(\gamma)\} \subseteq \Delta\Theta$, we have nothing to show if $b(D) = 0$, so suppose $b(D) > 0$.

Now, let $k' := k[1 - b(D)] \in [0, 1)$; let $\gamma' := \frac{1}{1-k'}\left[(1-k)\gamma + k\int_D \mu\,\mathrm{d}b(\mu)\right] \in \Delta\Theta$; and let $\beta' := \frac{1}{1-b(D)}\int_{(\Delta\Theta)\setminus D} \mu\,\mathrm{d}b(\mu)$ if $b(D) < 1$, and $\beta' := \mu_0$ if $b(D) = 1$. Because $k'\beta' + (1-k')\gamma' = k\beta + (1-k)\gamma$ and $(1-k')\gamma' \geq (1-k)\gamma$ by construction, (β', γ', k') is feasible in (2). In what follows, we show (β', γ', k') is an optimal solution to (2) with the desired features.

First, by construction, γ' is in the closed convex hull of $\{\bar{v} \geq \bar{v}(\gamma)\}$. But $\{\bar{v} \geq \bar{v}(\gamma)\}$ is closed and convex because $\bar{v}$ is upper semicontinuous and quasiconcave, implying $\bar{v}(\gamma') \geq \bar{v}(\gamma)$. If $k' = 0$ (in which case $\beta' = \gamma' = \mu_0$ by construction), this ranking implies $\bar{v}(\gamma') \geq (1-k)\bar{v}(\gamma) + k\hat{v}_{\wedge\gamma}(\beta)$, so that (β', γ', k') is optimal too, establishing the claim.

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We now focus on the remaining case in which $0 < k' < 1$. That $\bar{v}(\gamma') \geq \bar{v}(\gamma)$ implies $b' := \frac{1}{1-b(D)} b\left((\cdot) \cap D\right) \in \mathcal{P}(\beta')$ has $b'\{v < \bar{v}(\gamma')\} = 1$. Moreover,

$$\begin{aligned}
(1-k')\bar{v}(\gamma') + k'\hat{v}_{\wedge\gamma'}(\beta') &\geq (1-k')\bar{v}(\gamma') + k' \int v_{\wedge\gamma'} \, \mathrm{d}b' \\
&= \left[1 - k + k\beta(D)\right] \bar{v}(\gamma') + k' \int v_{\wedge\gamma'} \, \mathrm{d}b' \\
&= (1-k)\bar{v}(\gamma') + k \int v_{\wedge\gamma'} \, \mathrm{d}b \\
&\geq (1-k)\bar{v}(\gamma) + k \int v_{\wedge\gamma} \, \mathrm{d}b.
\end{aligned}$$

Optimality of (β, γ, k) in (2) then implies (β', γ', k') is optimal too. Therefore, the inequalities in the above chain must hold with equality, from which the first line of the above chain yields $\hat{v}_{\wedge\gamma'}(\beta') = \int v_{\wedge\gamma'} \, \mathrm{d}b'$. Thus, (β', γ', k') and b' are as required. □

Although our main purpose for the above lemma is to prove Corollary 1, note Lemma 4 can be useful in narrowing the search for a solution to Theorem 1's program. For example, in the context of the central bank example, the lemma immediately implies that (for any χ at which S can do strictly better than her no-credibility value) one optimally sets $\beta \leq \frac{1}{4}$.

We now proceed to prove the corollary. Our proof applies to the general model (not assuming A and Θ are finite, and not assuming $\boldsymbol{\chi}$ is state independent). Specifically, we show two things. First, some S-optimal $\boldsymbol{\chi}$-equilibrium entails no more than $|A|$ on-path messages. Second, if Θ is finite, some S-optimal $\boldsymbol{\chi}$-equilibrium entails no more than $2|\Theta| - 1$ on-path messages. The central-bank example, for which every S-optimal equilibrium requires at least three on-path messages when $2/3 < \chi < 3/4$, demonstrates both bounds are tight.

Proof of Corollary 1. By Lemma 4, some optimal solution (β, γ, k) to program (2) exists such that either (1) $(\beta, \gamma, k) = (\mu_0, \mu_0, 0)$ or (2) $k > 0$, and some $\tilde{b} \in \mathcal{P}(\beta)$ has $\int v_{\wedge\gamma} \, \mathrm{d}\tilde{b} = \hat{v}_{\wedge\gamma}(\beta)$ and $\tilde{b}\{v < \bar{v}(\gamma)\} = 1$. Let $s_i := \bar{v}(\gamma)$.

In case 1, we observe that some $g \in \mathcal{P}(\mu_0)$ exists with $g\{V \ni s_i\} = 1$ and $|\mathrm{supp}(g)|$ is weakly below the given cardinality bound. In case 2, we observe that some $b \in \mathcal{P}(\beta)$ and $g \in \mathcal{P}(\gamma)$ exist with $b\{v < s_i\} = g\{V \ni s_i\} = 1$, and $|\mathrm{supp}(b)| + |\mathrm{supp}(g)|$ is weakly below the given cardinality bound. In either case, the proof of Lemma 1 (applied with $b = g$ in case 1) yields an S-optimal equilibrium that respects the cardinality bound on on-path messages.

First, we prove the bound based on the number of actions. Letting $A_+ := \{a \in A : u_S(a) \geq s_i\}$, (the proof of) Proposition 2 from Lipnowski and Ravid (2020) delivers some $g \in \mathcal{P}(\gamma)$ such that $g\{V \ni s_i\} = 1$ and $|\text{supp}(g)| \leq |A_+|$. In case 1, nothing remains to be shown, so we now focus on case 2. Because $b \in \mathcal{P}(\beta)$ is such that $\text{argmax}_{a\in A} \int u_R(a, \cdot)\,\mathrm{d}\mu \subseteq A \backslash A_+$ a.s.-$b(\mu)$, (the proof of) Proposition 1 from Kamenica and Gentzkow (2011) delivers some $b \in \mathcal{P}(\beta)$ such that $|\text{supp}(b)| \leq |A \setminus A_+|$.[23] Hence, some S-optimal $\boldsymbol{\chi}$-equilibrium $(\xi, \sigma, \alpha, \pi)$ exists in which some measurable $M^* \subseteq M$ with $|M^*| \leq |A|$ has $\xi(M^*|\cdot) = \sigma(M^*|\cdot) = \mathbf{1}$.

Now, supposing $n := |\Theta| < \infty$, we prove the bound based on the number of states. Lemma 1 of Lipnowski and Ravid (2020) implies γ is in the convex hull of the compact set $\{V \ni s_i\}$, and then Caratheodory's theorem says γ is in the convex hull of some affinely independent subset $D \subseteq \{V \ni s_i\}$. Clearly, $|D| \leq n$, so nothing remains to be shown in case 1; let us now focus on case 2.

As $|D| < \infty$, we can without loss remove elements from D to ensure γ is a proper convex combination of all elements of D. By Choquet's theorem, $\tilde{b}$ is the barycenter of extreme points of $\mathcal{P}(b)$, which must then be solutions to $\max_{b\in\mathcal{P}(\beta)} \int v_{\wedge\gamma}\,\mathrm{d}b$. Taking one such extreme point yields $b \in \text{ext}\mathcal{P}(\beta)$ such that $b\{v < s_i\} = 1$ and $\int v_{\wedge\gamma}\,\mathrm{d}b = \hat{v}_{\wedge\gamma}(\beta)$. Because extreme points of $\mathcal{P}(\beta)$ have affinely independent support, it follows that $|\text{supp}(b)| \leq n$. Hence, some S-optimal $\boldsymbol{\chi}$-equilibrium $(\xi, \sigma, \alpha, \pi)$ exists in which some $M^* \subseteq M$ with $|M^*| \leq n + |D|$ has $\xi(M^*|\cdot) = \sigma(M^*|\cdot) = \mathbf{1}$. The corollary then follows if we can establish (in case 2) that $|D| < n$.

Assume for a contradiction that $|D| = n$. Then, the set of proper convex combinations of all elements of $|D|$ is an open subset of $\Delta\Theta$ that contains γ. In particular, some proper convex combination γ' of γ and μ_0 lies in the convex hull of $|D|$. Observe three properties of γ'. First, by construction, some $k' \in (0, k)$ exists such that $k'\beta + (1 - k')\gamma' = \mu_0$. Second, quasiconcavity of $\bar{v}$ implies $\bar{v}(\gamma') \geq \min \bar{v}(D) \geq s_i$. Third,

$$(1 - k')\gamma' = \mu_0 - k'\beta \geq \mu_0 - k\beta = (1 - k)\gamma,$$

[23] In both of the cited propositions, the result we use is proven in the cited paper, but not written in the proposition's statement. The proof of Proposition 2 from Lipnowski and Ravid (2020) shows any attainable equilibrium S payoff of the cheap-talk game is attainable in an equilibrium in which every on-path message is a pure-action recommendation, and the recommended action is S's preferred action in the support of R's (possibly mixed-action) response to that recommendation. The proof of Proposition 1 from Kamenica and Gentzkow (2011) shows, given a communication protocol with R best responding to Bayesian beliefs, that communication can be garbled to an incentive-compatible direct recommendation producing the same joint distribution of states and actions.

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so that (β, γ', k') is feasible in program (2). Hence,

$$k'\hat{v}_{\wedge\gamma'}(\beta) + (1-k')\bar{v}(\gamma') \geq k'\hat{v}_{\wedge\gamma}(\beta) + (1-k')s_i > k\hat{v}_{\wedge\gamma}(\beta) + (1-k)s_i,$$

contradicting the optimality of (β, γ, k). □

B.1.4 Further Consequences of Lemma 1 and Theorem 1

In this subsection, we record some properties of the $\boldsymbol{\chi}$-equilibrium payoff set and S's favorite $\boldsymbol{\chi}$-equilibrium payoff. We use these properties in the subsequent analysis.

Corollary 2. *The set of χ-equilibrium summaries (p, s_o, s_i) at prior μ_0 is a compact-valued, upper-hemicontinuous correspondence of (μ_0, χ) on $\Delta\Theta \times [0,1]$.*

Proof. Let Y_G be the graph of V and let Y_B be the graph of $[\min V, \max u_S(A)]$, both compact because V is a Kakutani correspondence.

Let X be the set of all $(\mu_0, p, g, b, \chi, k, s_o, s_i) \in (\Delta\Theta)\times(\Delta\Delta\Theta)^3\times[0,1]^2\times[\text{co } u_S(A)]^2$ such that

- $kb + (1-k)g = p$;
- $(1-\chi)\int_{\Delta\Theta} \mu \, \mathrm{d}g(\mu) + \chi \int_{\Delta\Theta} \mu \, \mathrm{d}b(\mu) = \mu_0$;
- $(1-k)\int_{\Delta\Theta} \mu \, \mathrm{d}g(\mu) \geq (1-\chi)\mu_0$;
- $g \otimes \delta_{s_i} \in \Delta(Y_G)$ and $b \otimes \delta_{s_i} \in \Delta(Y_B)$; and
- $k\int_{\Delta\Theta} \min V \, \mathrm{d}b \leq (k-\chi)\, s_i + \chi s_o \leq k\int_{\Delta\Theta} s_i \wedge v \, \mathrm{d}b$.

As an intersection of compact sets, X is itself compact. By Lemma 1, the equilibrium summary correspondence has a graph that is a projection of X, and so is itself compact. Therefore, it is compact valued and upper hemicontinuous.

□

Corollary 3. *For any $\mu_0 \in \Delta\Theta$, the map*

$$\{\boldsymbol{\chi} : \Theta \to [0,1] : \ \boldsymbol{\chi} \textit{ measurable}\} \to \mathbb{R}$$
$$\boldsymbol{\chi} \mapsto v^*_{\boldsymbol{\chi}}(\mu_0)$$

is weakly increasing.

Proof. This result follows immediately from Theorem 1 (the general version, with state-dependent credibility, proven above) because increasing credibility weakly expands the constraint set. □

Corollary 4. *For any $\mu_0 \in \Delta\Theta$, the map*

$$\begin{aligned}[0,1] &\to \mathbb{R} \\ \chi &\mapsto v^*_\chi(\mu_0)\end{aligned}$$

is weakly increasing and right-continuous.

Proof. That it is weakly increasing is a specialization of Corollary 3. That it is upper semicontinuous (and so, since nondecreasing, it is right-continuous) follows directly from Corollary 2. □

Corollary 5. *For any $\chi \in [0,1]$, the map $v^*_\chi : \Delta\Theta \to \mathbb{R}$ is upper semicontinuous.*

Proof. This result is immediate from Corollary 2. □

B.2 Varying Credibility: Proofs for Section 4

In this section, we provide proofs for the results reported in section 4. We note these results are stated for the version of the model developed in the main text (with finite action space, finite state space, and state-independent credibility). In contrast to the proof of Theorem 1, finiteness plays a nontrivial role in the proofs of these propositions. As our proofs make clear, the same results would hold with state-dependent credibility.

B.2.1 Productive Mistrust: Proof of Proposition 1

In this section, we prove Proposition 1 as stated in the main text. Whereas this proposition is stated for state-independent credibility, it immediately implies the following result for the case in which credibility is allowed to depend on the state:

Corollary 6. *Consider a finite and generic model in which S is not a two-faced SOB. Then, a full-support prior and state-dependent credibility levels $\boldsymbol{\chi}' < \boldsymbol{\chi}$ exist such that every S-optimal $\boldsymbol{\chi}'$ equilibrium is strictly better for R than every S-optimal $\boldsymbol{\chi}$-equilibrium.*

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As explained in the main text, one can divide the proof of Proposition 1 into two parts. The first part proves the proposition for the case in which Θ is binary. The second part uses a continuity argument to extend the binary-state result to any finite-state environment.

Productive Mistrust with Binary States We first verify our sufficient conditions for productive mistrust to occur in the binary-state world in the lemma below. In addition to being a special case of the proposition, it will also be an important lemma for proving the more general result.

To this end, to introducing a more detailed language for our key SOB condition is useful. Given a prior $\mu \in \Delta\Theta$, say S is **an SOB at** μ if every $p \in \mathcal{P}(\mu)$ is outperformed by an SOB policy $p' \in \mathcal{P}(\mu)$, that is, has $\int v \, \mathrm{d}p' \geq \int v \, \mathrm{d}p$.

Lemma 5. *Suppose $|\Theta| = 2$, the model is finite and generic, and a full-support belief $\mu \in \Delta\Theta$ exists such that the sender is not an SOB at μ. Then, a full-support prior μ_0 and credibility levels $\chi' < \chi$ exist such that every S-optimal χ'-equilibrium is both strictly better for R and more Blackwell-informative than every S-optimal χ-equilibrium.*

Moreover, some full-support belief μ_+ exists such that any solution (β, γ, k) to the program in Theorem 1 at prior μ_0 and credibility level in $\{\chi, \chi'\}$ has $\gamma = \mu_+$.

Proof. First, note the genericity assumption delivers full-support μ' such that $V(\mu') = \{\max v\, (\Delta\Theta)\}$.

Name our binary-state space $\{0, 1\}$ and identify $\Delta\Theta = [0, 1]$ in the obvious way. The function $v : [0, 1] \to \mathbb{R}$ is upper semicontinuous and piecewise constant, which implies its concavification v_1^* is piecewise affine. That is, some $n \in \mathbb{N}$ and $\{\mu^i\}_{i=0}^n$ exist such that $0 = \mu^0 \leq \cdots \leq \mu^n = 1$ and $v_1^*|_{[\mu^{i-1}, \mu^i]}$ is affine for every $i \in \{1, \ldots, n\}$. Taking n to be minimal, we can assume $\mu^0 < \cdots < \mu^n$ and the slope of $v_1^*|_{[\mu^{i-1}, \mu^i]}$ is strictly decreasing in i. Therefore, some $i_0, i_1 \in \{0, \ldots, n\}$ exist such that $i_1 \in \{i_0, i_0 + 1\}$ and $\operatorname{argmax}_{\tilde{\mu} \in [0,1]} v_1^*(\tilde{\mu}) = [\mu^{i_0}, \mu^{i_1}]$. That the sender is not an SOB at μ implies $i_0 > 1$ or $i_1 < n - 1$. Without loss of generality, say $i_0 > 1$. Now let $\mu_- := \mu^{i_0 - 1}$ and $\mu_+ := \mu^{i_0}$.

We now find a $\mu_0 \in (\mu_-, \mu_+)$ such that $\bar{v}|_{[\mu_0, \mu_+)}$ is constant and lies strictly below $v_1^*|_{[\mu_0, \mu_+)}$. To do so, recall the model is finite, and so $\bar{v}$ has a finite range and is piecewise constant. It follows some $\epsilon > 0$ exists such that $\bar{v}$ is constant on $(\mu_+ - \epsilon, \mu_+)$. Because $v_1^* \colon [0, 1] \to \mathbb{R}$ is concave and upper semicontinuous, it is in fact continuous, and so

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admits an $\tilde{\epsilon} \in (0, \mu_+)$ such that every $\tilde{\mu} \in (\mu_+ - \tilde{\epsilon}, \mu_+)$ has

$$v_1^*(\tilde{\mu}) > \max\left[\bar{v}([0,1]) \setminus \{\max \bar{v}([0,1])\}\right] \geq \bar{v}(\tilde{\mu}),$$

where the last inequality follows from $\bar{v}|_{[0,\mu_+)} \leq v_1^*|_{[0,\mu_+)} < v_1^*(\mu_+)$. Thus, the desired properties are satisfied by any $\mu_0 \in (\max\{\mu_-, \mu_+ - \epsilon, \mu_+ - \tilde{\epsilon}\}, \mu_+)$. Let μ_0 be one such belief.

To summarize, the beliefs $\mu_-, \mu_0, \mu_+ \in [0,1]$ are such that $0 < \mu_- < \mu_0 < \mu_+$; $\hat{v}_{\wedge\mu_+} = \hat{v} = v_1^*$ is affine on $[\mu_-, \mu_+]$ and on no larger interval; $\hat{v}_{\wedge\mu_+}$ is strictly increasing on $[0, \mu_+]$; $v_0^* = \bar{v}$ is constant on $[\mu_0, \mu_+)$.

Let $\chi \in [0,1]$ be the smallest credibility level such that $v_\chi^*(\mu_0) = v_1^*(\mu_0)$, which exists by Corollary 4. That $v_0^*(\mu_0) < v_1^*(\mu_0)$ implies $\chi > 0$. Notice μ_+ has full support, because $0 \leq \mu_- < \mu_+ \leq \mu' < 1$. It follows that $\chi < 1$. Consider now the following claim.

Claim: *Given $\chi' \in [0, \chi]$, suppose*

$$(\beta', \gamma', k') \in \operatorname{argmax}_{(\beta,\gamma,k)\in[0,1]^3} \left\{ k\hat{v}_{\wedge\gamma}(\beta) + (1-k)\bar{v}(\gamma) \right\} \tag{3}$$
$$\text{s.t.} \qquad k\beta + (1-k)\gamma = \mu_0, \;\; (1-k)(\gamma, 1-\gamma) \geq (1-\chi')(\mu_0, 1-\mu_0),$$

and the objective attains a value strictly higher than $\bar{v}(\mu_0)$. Then,

- *$\gamma' = \mu_+$ and $\beta' \leq \mu_-$.*
- *If $b' \in \mathcal{P}(\beta')$ and $g' \in \mathcal{P}(\gamma')$ are such that $p' = k'b' + (1-k')g'$ is the information policy of an S-optimal χ'-equilibrium, then $b'[0, \mu_-] = g'\{\mu_+\} = 1$.*

We now prove the claim.

Suppose first $\gamma' > \mu_+$ for a contradiction, and let $k'' > 0$ be the unique solution to $k''\beta' + (1-k'')\mu_+ = \mu_0$. Observe $k'' < k'$, and so

$$\begin{aligned}(1-k'')(\mu_+, 1-\mu_+) &= (\mu_0, 1-\mu_0) - k''(\beta', 1-\beta') \\ &\geq (\mu_0, 1-\mu_0) - k'(\beta', 1-\beta') \\ &= (1-k')(\gamma', 1-\gamma') \geq (1-\chi')(\mu_0, 1-\mu_0).\end{aligned}$$

Because

$$k''\hat{v}_{\wedge\mu_+}(\beta') + (1-k'')\bar{v}(\mu_+) \geq k''\hat{v}_{\wedge\gamma'}(\beta') + (1-k'')\bar{v}(\gamma') > k'\hat{v}_{\wedge\gamma'}(\beta') + (1-k')\bar{v}(\gamma'),$$

(β', μ_+, k'') is a feasible solution that would strictly outperform (β', γ', k'), contradicting optimality of (β', γ', k'). It follows $\gamma' \leq \mu_+$.

Next, note $\bar{v}$—as a weakly quasiconcave function that is nondecreasing and nonconstant over $[\mu_0, \mu_+]$—is nondecreasing over $[0, \mu_+]$. Moreover, $\lim_{\mu \nearrow \mu_+} \bar{v}(\mu) = \bar{v}(\mu_0) < \bar{v}(\mu_+)$. Therefore, if $\gamma' < \mu_+$, it would follow that $k'\hat{v}_{\wedge\gamma'}(\beta') + (1-k')\bar{v}(\gamma') \leq \bar{v}(\gamma') \leq \bar{v}(\mu_0)$. Given the hypothesis that (β', γ', k') strictly outperforms $\bar{v}(\mu_0)$, it follows that $\gamma' = \mu_+$. A direct implication is that

$$(\beta', k') \in \text{argmax}_{(\beta,k)\in[0,1]^2} \left\{ k\hat{v}_{\wedge\mu_+}(\beta) + (1-k)\max v[0,\mu_+] \right\}$$
$$\text{s.t.} \qquad k\beta + (1-k)\mu_+ = \mu_0, \ (1-k)(1-\mu_+) \geq (1-\chi')(1-\mu_0).$$

Let us now see why we cannot have $\beta' \in (\mu_-, \mu_0)$. Because $\hat{v}_{\wedge\mu_+}$ is affine on $[\mu_+, \mu_-]$, replacing such (k', β') with (k, μ_-) that satisfies $k\mu_- + (1-k)\mu_+ = \mu_0$ necessarily has $(1-k)(\mu_+, 1-\mu_+) \gg (1-\chi')(\mu_0, 1-\mu_0)$. This would contradict minimality of χ. Therefore, $\beta' \leq \mu_-$.

We now prove the second bullet. First, every $\mu < \mu_+$ satisfies $v(\mu) \leq v_1^*(\mu) < v_1^*(\mu_+) = v(\mu_+)$. This property implies δ_{μ_+} is the unique $g \in \mathcal{P}(\mu_+)$ with $\inf v(\text{supp} g) \geq v(\mu_+)$. Therefore, $g' = \delta_{\mu_+}$. Second, the measure $b' \in \mathcal{P}(\beta')$ can be expressed as $b' = (1-\lambda)b_L + \lambda b_R$ for $b_L \in \Delta[0, \mu_-]$, $b_R \in \Delta(\mu_-, 1]$, and $\lambda \in [0, 1)$. Note $(\mu_-, v(\mu_-))$ is an extreme point of the subgraph of v_1^*, and therefore an extreme point of the subgraph of $\hat{v}_{\wedge\mu_+}$. Taking the unique $\hat{\lambda} \in [0, \lambda]$ such that $\hat{b} := (1-\hat{\lambda})b_L + \hat{\lambda}\delta_{\mu_-} \in \mathcal{P}(\beta')$, it follows that $\int_{[0,1]} \hat{v}_{\wedge\mu_+} \, d\hat{b} \geq \int_{[0,1]} \hat{v}_{\wedge\mu_+} \, db'$, strictly so if $\hat{\lambda} < \lambda$. But $\hat{\lambda} < \lambda$ necessarily if $\lambda > 0$, because $\int_{[0,1]} \mu \, d\beta_R(\mu) > \mu_-$. Optimality of b' then implies $\lambda = 0$, that is, $b'[0, \mu_-] = 1$. This observation completes the proof of the claim.

With the claim in hand, we can now prove the lemma. The claim implies that, for credibility level χ, any solution (β^*, γ^*, k^*) of the program (3) is such that $\gamma^* = \mu_+$, $k^* = \frac{\mu_+ - \mu_0}{\mu_+ - \beta^*}$, and β^* solves

$$\max_{\beta\in[0,\mu_-]} \left\{ \frac{\mu_+ - \mu_0}{\mu_+ - \beta} \hat{v}_{\wedge\mu_+}(\beta) + \frac{\mu_0 - \beta}{\mu_+ - \beta} \bar{v}(\mu_+) \right\}.$$

Note that because $\bar{v}(\mu_+) = v(\mu_+) = \hat{v}_{\wedge\mu_+}(\mu_+)$, any $\beta \in [0, \mu_-]$ has

$$\frac{\mu_+ - \mu_0}{\mu_+ - \beta} \hat{v}_{\wedge\mu_+}(\beta) + \frac{\mu_0 - \beta}{\mu_+ - \beta} \hat{v}_{\wedge\mu_+}(\mu_+) \leq \hat{v}_{\wedge\mu_+} \left(\frac{\mu_+ - \mu_0}{\mu_+ - \beta}\beta + \frac{\mu_0 - \beta}{\mu_+ - \beta}\mu_+ \right) = \hat{v}_{\wedge\mu_+}(\mu_0)$$

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by concavity of $\hat{v}_{\wedge\mu_+}$. Moreover, the inequality is strict for $\beta < \mu_-$ but holds with equality for $\beta = \mu_-$, because $\hat{v}_{\wedge\mu_+}$ is affine on $[\mu_-, \mu_+]$ and on no larger interval. Hence, the unique solution to (3) is (μ_-, μ_+, k^*), where $k^*\mu_- + (1-k^*)\mu_+ = \mu_0$. Moreover, the minimality property defining χ implies $(1-k^*)(1-\mu_+) = (1-\chi)(1-\mu_0)$.

Given $\chi' < \chi$ sufficiently close to χ, one can verify directly that (β', μ_+, k') is feasible, where

$$k' := 1 - \tfrac{1-\chi'}{1-\chi}(1-k^*) \text{ and } \beta' := \tfrac{1}{k'}\left[\mu_0 - (1-k')\mu_+\right].$$

Because $\hat{v}_{\wedge\mu_+}$ is a continuous function, it follows that $v^*_{\chi'}(\mu_0) \nearrow v^*_{\chi}(\mu_0)$ as $\chi' \nearrow \chi$. In particular, $v^*_{\chi'}(\mu_0) > v^*_0(\mu_0)$ for $\chi' < \chi$ sufficiently close to χ. Fix such a χ'.

Let p' be any S-optimal χ'-equilibrium information policy. Appealing to the claim, some $b' \in \mathcal{P}(\beta') \cap \Delta[0, \mu_-]$ exists such that $p' \in \text{co}\{b', \delta_{\mu_+}\}$. Therefore, p' is weakly more Blackwell-informative than p^*. Finally, because $(1-k^*)(1-\mu_+) = (1-\chi)(1-\mu_0)$ and $\chi' < \chi$, feasibility of p' tells us $p' \neq p^*$. Therefore (the Blackwell order being antisymmetric), p' is strictly more informative than p^*.

All that remains is to show the receiver's optimal payoff is strictly higher given p' than given p^*. To that end, fix sender-preferred receiver best responses a_- and a_+ to μ_- and μ_+, respectively. Because the receiver's optimal value given p^* is attainable using only actions $\{a_-, a_+\}$, and the same value is feasible given only information p' and using only actions $\{a_-, a_+\}$, it suffices to show that there are beliefs in the support of p' to which neither of $\{a_-, a_+\}$ is a receiver best response. But, every $\mu \in [0, \mu_-)$ satisfies

$$v(\mu) \leq \bar{v}(\mu) < \bar{v}(\mu_-) = \min\{\bar{v}(\mu_-), \bar{v}(\mu_+)\};$$

that is, $\max u_S\left(\text{argmax}_{a\in A}\, u_R(a,\mu)\right) < \min\{u_S(a_-), u_S(a_+)\}$. The result follows. $\square$

Productive Mistrust with Many States: Proof of Proposition 1 Given Lemma 5, we need only prove the proposition for the case of $|\Theta| > 2$, which we do below. The proof intuition is as follows. Using the binary-state logic, one can always obtain a binary-support prior μ_0^∞ and credibility levels $\chi' < \chi$ such that R strictly prefers every S-optimal χ'-equilibrium to every S-optimal χ-equilibrium. We then find an interior direction through which to approach μ_0^∞, while keeping S's optimal equilibrium value under both credibility levels continuous. Genericity ensures such a direction exists despite $\bar{v}$ being discontinuous. The continuity in S's value from the identified direction then ensures upper hemicontinuity of S's optimal equilibrium policy set; that is, the limit of every sequence of S-optimal equilibrium policies from said direction must also be optimal under μ_0^∞. Now, if the proposition were false, one could construct a convergent sequence of S-optimal equilibrium policies from said direction for each credibility level, $\{p_n^\chi, p_n^{\chi'}\}_{n\geq 0}$, such that R would weakly prefer p_n^χ to $p_n^{\chi'}$. Because R's payoffs are continuous, R being weakly better off under χ than under χ' along the sequences would imply the same at the sequences' limits. Notice, though, such limits must be S-optimal for the prior μ_0^∞ by the choice of direction, meaning productive mistrust fails at μ_0^∞; that is, we have a contradiction. Below, we proceed with the formal proof.

Proof. Suppose some prior with binary support $\Theta_2 = \{\theta_1, \theta_2\}$ exists at which S is not an SOB. Let $\bar{s} := \max v(\Delta\Theta_2)$, and define the R value function $v_R\colon \Delta\Delta\Theta \to \mathbb{R}$ via $v_R(p) := \int_{\Delta\Theta} \max_{a\in A} u_R(a,\mu)\,\mathrm{d}p(\mu)$. Lemma 5 delivers some $\mu_0^\infty \in \Delta\Theta$ with support Θ_2 and credibility levels $\chi'' < \chi'$ such that every S-optimal χ''-equilibrium is strictly better for R than every S-optimal χ'-equilibrium. Consider the following claim.
<u>Claim:</u> Some sequence $\{\mu_0^n\}$ of full-support priors exists that converges to μ_0^∞ with

$$\liminf_{n\to\infty} v_\chi^*(\mu_0^n) \geq v_\chi^*(\mu_0^\infty) \text{ for } \chi \in \{\chi', \chi''\}.$$

Before proving the claim, let us argue it implies the proposition. Given the claim, assume for contradiction that for every $n \in \mathbb{N}$, prior μ_0^n admits some S-optimal χ'-equilibrium and χ''-equilibrium, $\Psi_n' = (p_n', s_{in}', s_{on}')$ and $\Psi_n'' = (p_n'', s''_{i\,n}, s''_{on})$, respectively, such that $v_R(p_n') \geq v_R(p_n'')$. Dropping to a subsequence if necessary, we may assume by compactness that $(\Psi'_n)_n$ and $(\Psi''_n)_n$ converge (in $\Delta\Delta\Theta \times [\mathrm{co}\, u_S(A)]^2$) to some $\Psi' = (p', s_i', s_o')$ and $\Psi'' = (p'', s_i'', s_o'')$, respectively. By Corollary 2, for every credibility level χ, the set of χ-equilibria is an upper-hemicontinuous correspondence of the prior. Therefore, Ψ' and Ψ'' are χ'- and χ''-equilibria, respectively, at prior μ_0^∞. Continuity of v_R (by Berge's theorem) then implies $v_R(p') \geq v_R(p'')$. Finally, by the claim, it must be that Ψ' and Ψ'' are S-optimal χ'- and χ''-equilibria, respectively, contradicting the definition of μ_0^∞. Therefore, some $n \in \mathbb{N}$ exists such that the full-support prior μ_0^n is as required for the proposition.

So all that remains is to prove the claim, which we do by constructing the desired sequence.

First, Lemma 5 delivers some $\gamma^\infty \in \Delta\Theta$ with support Θ_2 such that $\bar{v}(\gamma^\infty) = \bar{s}$ and, for $\chi \in \{\chi', \chi''\}$, any solution (β, γ, k) to the program in Theorem 1 at prior μ_0^∞ and credibility level χ has $\gamma = \gamma^\infty$.

Let us now show a closed convex set $D \subseteq \Delta\Theta$ exists that contains γ^∞, has a nonempty interior, and satisfies $\bar{v}|_D = \bar{s}$. Notice, first, that the genericity assumption delivers μ' with support Θ_2 such that $V(\mu') = \{\bar{s}\}$. Then, for any $n \in \mathbb{N}$, let $B_n \subseteq \Delta\Theta$ be the closed ball (say, with respect to the Euclidean metric) of radius $\frac{1}{n}$ around μ', and let $D_n := \text{co}\left[\{\gamma^\infty\} \cup B_n\right]$. Because $v|_{\Delta\Theta_2} \leq \bar{s}$ and $\bar{v} = \max_{p\in\mathcal{P}(\cdot)} \inf v(\text{supp}(p))$ (see Lipnowski and Ravid 2020, Theorem 2), it follows $\bar{v}|_{\Delta\Theta_2} \leq \bar{s}$ as well. Because V is upper hemicontinuous, the hypothesis on μ' ensures $\bar{v}|_{B_n} \geq v|_{B_n} = \bar{s}$ for sufficiently large $n \in \mathbb{N}$; quasiconcavity then tells us $\bar{v}|_{D_n} \geq \bar{s}$. Assume now, for a contradiction, that every $n \in \mathbb{N}$ has $\bar{v}|_{D_n} \not\leq \bar{s}$. That is, each $n \in \mathbb{N}$ admits some $\lambda_n \in [0,1]$ and $\mu'_n \in B_n$ such that $\bar{v}\left((1-\lambda_n)\gamma^\infty + \lambda_n\mu'_n\right) > \bar{s}$. In this case, each $n \in \mathbb{N}$ has $\bar{v}\left((1-\lambda_n)\gamma^\infty + \lambda_n\mu'_n\right) \geq \hat{s} := \min\left[\bar{v}(\Delta\Theta) \cap (\bar{s}, \infty)\right]$ (observe $\hat{s}$ is well defined because $|\bar{v}(\Delta\Theta)| < \infty$ due to the model being finite). Dropping to a subsequence, we get a strictly increasing sequence $(n_\ell)_{\ell=1}^\infty$ of natural numbers such that (because $[0,1]$ is compact) $\lambda_{n_\ell} \xrightarrow{\ell\to\infty} \lambda \in [0,1]$ and $\bar{v}\left((1-\lambda_{n_\ell})\gamma^\infty + \lambda_{n_\ell}\mu'_{n_\ell}\right) \geq \hat{s}$ for every $\ell \in \mathbb{N}$. Because $\bar{v}$ is upper semicontinuous, the sequence of inequalities would imply $\bar{v}\left((1-\lambda)\gamma^\infty + \lambda\mu'\right) \geq \hat{s} > \bar{s}$, contradicting the definition of $\bar{s}$ and μ'. Therefore, some $D \in \{D_{n_\ell}\}_{\ell=1}^\infty$ is as desired.

In what follows, let $\gamma_1 \in D$ be some interior element with full support. Then, for each $n \in \mathbb{N}$, define $\mu_0^n := \frac{n-1}{n}\mu_0^\infty + \frac{1}{n}\gamma_1$. We show the sequence $(\mu_0^n)_{n=1}^\infty$—a sequence of full-support priors converging to μ_0^∞—is as desired. To that end, fix $\chi \in \{\chi', \chi''\}$ and some $(\beta, k) \in \Delta\Theta \times [0,1]$ such that $(\beta, \gamma^\infty, k)$ solves the program in Theorem 1 at prior μ_0^∞. Then, for any $n \in \mathbb{N}$, let

$$\begin{aligned}
\epsilon_n &:= \tfrac{1}{n-(n-1)k} \in (0,1], \\
\gamma_n &:= (1-\epsilon_n)\gamma^\infty + \epsilon_n\gamma_1 \in D, \\
k_n &:= \tfrac{n-1}{n}k \in [0,k).
\end{aligned}$$

Given these definitions,

$$\begin{aligned}
(1-k_n)\gamma_n &= \tfrac{1}{n}\left[n-(n-1)k\right]\gamma_n \\
&= \tfrac{1}{n}\left\{\left[n-(n-1)k-1\right]\gamma^\infty + \gamma_1\right\} \\
&= \tfrac{n-1}{n}(1-k)\gamma^\infty + \tfrac{1}{n}\gamma_1 \\
&\geq \tfrac{n-1}{n}(1-\chi)\mu_0^\infty + \tfrac{1}{n}\gamma_1 \geq (1-\chi)\mu_0^n,
\end{aligned}$$

and

$$\begin{aligned} k_n\beta + (1-k_n)\gamma_n &= \tfrac{n-1}{n}k\beta + \tfrac{n-1}{n}(1-k)\gamma^\infty + \tfrac{1}{n}\gamma_1 \\ &= \tfrac{n-1}{n}\mu_0^\infty + \tfrac{1}{n}\gamma_1 = \mu_0^n. \end{aligned}$$

Therefore, (β, γ_n, k_n) is χ-feasible at prior μ_0^n. As a result,

$$\begin{aligned} v_\chi^*(\mu_0^n) &\geq k_n\hat{v}_{\wedge\gamma_n}(\beta) + (1-k_n)\bar{v}(\gamma_n) \\ &= k_n\hat{v}_{\wedge\gamma}(\beta) + (1-k_n)\bar{v}(\gamma) \text{ (since } \bar{v}(\gamma_n) = u) \\ &\xrightarrow{n\to\infty} k\hat{v}_{\wedge\gamma}(\beta) + (1-k)\bar{v}(\gamma) = v_\chi^*(\mu_0^\infty). \end{aligned}$$

This proves the claim, and hence the proposition. □

B.2.2 Collapse of Trust: Proof of Proposition 2

Proof. Let us establish a four-way equivalence between the three conditions in the proposition's statement and the following state-dependent-credibility analogue of condition (i):

$(i)'$ Every $\boldsymbol{\chi} \in [0,1]^\Theta$ and full-support prior μ_0 have $\lim_{\boldsymbol{\chi}'\nearrow\boldsymbol{\chi}} v_{\boldsymbol{\chi}'}^*(\mu_0) = v_{\boldsymbol{\chi}}^*(\mu_0)$, where convergence of $\boldsymbol{\chi}'(\cdot) \to \boldsymbol{\chi}(\cdot)$ is in the Euclidean topology on $\mathbb{R}^\Theta$.

Three of four implications are easy given Corollary 3. First, $(i)'$ trivially implies (i). Second ((iii) implies (ii)), in the absence of conflict, Lemma 1 from Lipnowski and Ravid (2020) tells us a 0-equilibrium exists with full information that generates sender value $\max v(\Delta\Theta) \geq v_1^*$; in particular, $v_0^* = v_1^*$. Third ((ii) implies $(i)'$), if $v_0^* = v_1^*$, Corollary 3 implies $v_{\boldsymbol{\chi}}^*$ is constant in $\boldsymbol{\chi}$, ruling out a collapse of trust (even under state-dependent credibility). Below, we show that any conflict implies a collapse of trust; that is, a failure of (iii) implies a failure of (i).

Suppose a conflict exists; that is, $\min_{\theta\in\Theta} v(\delta_\theta) < \max v(\Delta\Theta)$ or, equivalently, $\min_{\theta\in\Theta} \bar{v}(\delta_\theta) < \max \bar{v}(\Delta\Theta)$. Taking a positive affine transformation of u_S, we may assume without loss that $\min \bar{v}(\Delta\Theta) = 0$ and (because $\bar{v}(\Delta\Theta) \subseteq u_S(A)$ is finite) $\min[\bar{v}(\Delta\Theta) \setminus \{0\}] = 1$. The set $D := \arg\min_{\mu\in\Delta\Theta} \bar{v}(\mu) = \bar{v}^{-1}(-\infty, 1)$ is then open and nonempty. We can then consider some full-support prior $\mu_0 \in D$. For any scalar $\hat{\chi} \in [0,1]$, let

$$\Gamma(\hat{\chi}) := \{(\beta, \gamma, k) \in \Delta\Theta \times (\Delta\Theta \setminus D) \times [0,1] \colon k\beta + (1-k)\gamma = \mu_0,\ (1-k)\gamma \geq (1-\hat{\chi})\mu_0\},$$

and let $K(\hat{\chi})$ be its projection onto its last coordinate. Because the correspondence Γ is upper hemicontinuous and increasing (with respect to set containment), K inherits the same properties. Next, note $K(1) \ni 1$ (because $\bar{v}$ is nonconstant by the hypothesis that a conflict exists, so that $\Delta\Theta \neq D$) and $K(0) = \varnothing$ (as $\mu_0 \in D$). Therefore, $\chi := \min\{\hat{\chi} \in [0,1] : K(\hat{\chi}) \neq \varnothing\}$ exists and belongs to $(0,1]$.

Given any scalar $\chi' \in [0,\chi)$, it must be that $K(\chi') = \varnothing$. That is, if $\beta, \gamma \in \Delta\Theta$ and $k \in [0,1]$ with $k\beta + (1-k)\gamma = \mu_0$ and $(1-k)\gamma \geq (1-\chi')\mu_0$, then $\gamma \in D$. Thus, by Theorem 1, $v^*_{\chi'}(\mu_0) = \bar{v}(\mu_0) = 0$. There is, however, some $k \in K(\chi)$. By Theorem 1 and the definition of Γ, a χ-equilibrium generating ex-ante sender payoff of at least $k \cdot 0 + (1-k) \cdot 1 = (1-k) \geq (1-\chi)$ therefore exists. If $\chi < 1$, a collapse of trust occurs at credibility level χ.

The only remaining case is the one in which $\chi = 1$. In this case, some $\epsilon \in (0,1)$ and $\mu \in \Delta\Theta \setminus D$ exist such that $\epsilon\mu \leq \mu_0$. Then,

$$v^*_\chi(\mu_0) \geq \epsilon\bar{v}(\mu) + (1-\epsilon)\bar{v}\left(\tfrac{\mu_0 - \epsilon\mu}{1-\epsilon}\right) \geq \epsilon.$$

So, again, a collapse of trust occurs at credibility level χ. □

B.2.3 Robustness: Proof of Proposition 3

Before proving the proposition, let us briefly observe that the proposition as stated is equivalent to the analogous statement for state-dependent credibility. Indeed, given Corollary 3, any prior μ_0 and state-dependent credibility $\boldsymbol{\chi}$ has $v^*_\chi(\mu_0) \leq v^*_{\boldsymbol{\chi}}(\mu_0) \leq v^*_1(\mu_0)$ for $\chi = \min_{\theta\in\Theta} \boldsymbol{\chi}(\theta) \in [0,1]$. It follows immediately that $\lim_{\chi\nearrow 1} v^*_\chi(\mu_0) = v^*_1(\mu_0)$ if and only if $\lim_{\boldsymbol{\chi}\nearrow \mathbf{1}} v^*_{\boldsymbol{\chi}}(\mu_0) = v^*_{\mathbf{1}}(\mu_0)$, where convergence of $\boldsymbol{\chi} \to \mathbf{1}$ is in the Euclidean topology on $\mathbb{R}^\Theta$. That is, the stronger property of robustness of the commitment value to small *state-dependent* departures from perfect credibility is equivalent to that stated in the proposition.

We now proceed to proving the proposition for the case of state-independent credibility.

Proof. By Lipnowski and Ravid (2020, Lemma 1 and Theorem 2), S receives the benefit of the doubt (i.e., every $\theta \in \Theta$ is in the support of some member of $\text{argmax}_{\mu\in\Delta\Theta} v(\mu)$) if and only if some full-support $\gamma \in \Delta\Theta$ exists such that $\bar{v}(\gamma) = \max v(\Delta\Theta)$.

First, given a full-support prior μ_0, suppose $\gamma \in \Delta\Theta$ is full-support with $\bar{v}(\gamma) = \max v(\Delta\Theta)$. It follows immediately that $\hat{v}_{\wedge\gamma} = \hat{v} = v^*_1$. Let $r_0 := \min_{\theta\in\Theta} \frac{\mu_0\{\theta\}}{\gamma\{\theta\}} \in (0,\infty)$

and $r_1 := \max_{\theta\in\Theta} \frac{\mu_0\{\theta\}}{\gamma\{\theta\}} \in [r_0, \infty)$. Then, Theorem 1 tells us that for $\chi \in \left[\frac{r_1-r_0}{r_1}, 1\right)$,

$$\begin{aligned}
v^*_\chi(\mu_0) \geq & \sup_{\beta\in\Delta\Theta,\ k\in[0,1]} \left\{kv^*_1(\beta) + (1-k)v(\gamma)\right\} \\
& \quad \text{s.t.} \quad k\beta + (1-k)\gamma = \mu_0,\ (1-k)\gamma \geq (1-\chi)\mu_0 \\
= & \sup_{k\in[0,1]} \left\{kv^*_1\left(\tfrac{\mu_0-(1-k)\gamma}{k}\right) + (1-k)v(\gamma)\right\} \\
& \quad \text{s.t.} \quad (1-\chi)\mu_0 \leq (1-k)\gamma \leq \mu_0 \\
\geq & \sup_{k\in[0,1]} \left\{kv^*_1\left(\tfrac{\mu_0-(1-k)\gamma}{k}\right) + (1-k)v(\gamma)\right\} \\
& \quad \text{s.t.} \quad (1-\chi)r_1 \leq (1-k) \leq r_0 \\
\geq & \sup_{k\in[0,1]} \left\{kv^*_1\left(\tfrac{\mu_0-(1-k)\gamma}{k}\right) + (1-k)v(\gamma)\right\} \\
& \quad \text{s.t.} \quad (1-\chi)r_1 = (1-k) \\
= & \left[1-(1-\chi)r_1\right] v^*_1\left(\tfrac{\mu_0-(1-\chi)r_1\gamma}{1-(1-\chi)r_1}\right) + (1-\chi)r_1 v(\gamma).
\end{aligned}$$

But note v^*_1, being a concave function on a finite-dimensional space, is continuous on the interior of its domain. Therefore, $v^*_1\left(\frac{\mu_0-(1-\chi)r_1\gamma}{1-(1-\chi)r_1}\right) \to v^*_1(\mu_0)$ as $\chi \to 1$, implying $\liminf_{\chi\nearrow 1} v^*_\chi(\mu_0) \geq v^*_1(\mu_0)$. Finally, monotonicity of $\chi \mapsto v^*_\chi(\mu_0)$ implies $v^*_\chi(\mu_0) \to v^*_1(\mu_0)$ as $\chi \to 1$. That is, persuasion is robust to limited commitment.

Conversely, suppose S does not receive the benefit of the doubt (which of course implies v is nonconstant). Taking an affine transformation of u_S, we may assume without loss that $\max v(\Delta\Theta) = 1$ and (because $v(\Delta\Theta) \subseteq u_S(A)$ is finite) $\max[\bar{v}(\Delta\Theta) \setminus \{1\}] = 0$. Fix any full-support prior μ_0 and consider any credibility level $\chi \in [0,1)$. For any $\beta, \gamma \in \Delta\Theta,\ k \in [0,1]$ with $k\beta + (1-k)\gamma = \mu_0$ and $(1-k)\gamma \geq (1-\chi)\mu_0$, that S does not get the benefit of the doubt implies (see Lipnowski and Ravid, 2020, Theorem 1) that $\bar{v}(\gamma) \leq 0$, and therefore that $k\hat{v}_{\wedge\gamma}(\beta) + (1-k)v(\gamma) \leq 0$. Theorem 1 then implies $v^*_\chi(\mu_0) \leq 0$.

Fix some full-support $\mu_1 \in \Delta\Theta$ and some $\gamma \in \Delta\Theta$ with $v(\gamma) = 1$. For any $\epsilon \in (0,1)$, the prior $\mu_\epsilon := (1-\epsilon)\gamma + \epsilon\mu_1$ has full support and satisfies

$$v^*_1(\mu_\epsilon) \geq (1-\epsilon)v(\gamma) + \epsilon v(\mu_1) \geq (1-\epsilon) + \epsilon \cdot \min v(\Delta\Theta),$$

which is strictly positive for sufficiently small ϵ. Persuasion is therefore not robust to

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limited commitment at prior μ_ϵ. □

C Extension on Signaling Credibility

In this section, we consider the modified version of our model in which S learns her credibility type before announcing the official reporting protocol. By letting S commission a different official report based on her credibility, the modified model allows S to signal whether she can influence the report's message. We show such signaling has no impact on S's attainable payoffs. More precisely, every interim S-payoff profile (i.e., every pair specifying S's payoffs conditional on each credibility type) is attainable in a pooling equilibrium in which both credibility types choose the same official experiment. It follows that pooling equilibria are without loss as far as S payoffs are concerned. We also show an S-payoff profile is attainable in a pooling equilibrium if and only if it is attainable in a χ-equilibrium. Our definition will make the fact that every pooling-equilibrium payoff profile is attainable in a χ-equilibrium immediate: a pooling equilibrium of the modified game requires the same conditions as a χ-equilibrium, except S must also be willing to announce the equilibrium experiment conditional on her credibility type. For the converse direction, we show every χ-equilibrium can be implemented as a pooling equilibrium of the signaling game by appropriately constructing R's behavior off path. Thus, we show a three-way equivalence between S's payoffs in all equilibria of the signaling game, all pooling equilibria of the signaling game, and χ-equilibria of the original game. It follows that informing S of her ability to influence the report before its announcement has no impact on S's achievable payoffs.

C.1 On S's Equilibrium Payoff Sets

We begin by providing results on the space of S payoffs that will be of use in the extension that follows and may be of independent use. We return to the general specification of our model in which the state and action spaces may be finite or infinite, and the credibility level may or may not depend on the payoff state.

First, we characterize the set of payoffs attainable in a $\boldsymbol{\chi}$-equilibrium by an influencing S, in particular showing this payoff set is an interval. Then, we show the set of ex-ante S payoffs attainable in a $\boldsymbol{\chi}$-equilibrium is an interval as well.

Toward the proof, we first record a useful property of Kakutani correspondences.

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Fact 1. *The range of a Kakutani correspondence from a nonempty, compact, convex space to $\mathbb{R}$ is a nonempty compact interval.*

Proof. Nonemptiness is trivial. Compactness of the range holds because the correspondence is upper hemicontinuous on a compact domain. Convexity follows from the intermediate value theorem for correspondences (e.g., Lemma 2 of de Clippel, 2008). □

Next, we establish convexity and compactness of the sets of S's possible $\boldsymbol{\chi}$-equilibrium ex-ante payoffs and payoffs from influencing. To do so, we now provide a characterization of the set

$$S_i^{\boldsymbol{\chi}} := \{s_i \in \mathbb{R} \colon\ (p, s_o, s_i) \text{ is a } \chi\text{-equilibrium summary for some } p, s_o\}.$$

Lemma 6. *Let $s_i \in \mathbb{R}$. Then $s_i \in S_i^{\boldsymbol{\chi}}$ if and only if some $k \in [0,1], \gamma, \beta \in \Delta\Theta$ exist such that*

(i) $k\beta + (1-k)\gamma = \mu_0$,

(ii) $(1-k)\gamma \geqslant (\mathbf{1}-\boldsymbol{\chi})\mu_0$,

(iii) $\max\{\underline{w}(\beta), \underline{w}(\gamma)\} \leq s_i \leq \bar{v}(\gamma)$.

Moreover, the set $S_i^{\boldsymbol{\chi}}$ is a nonempty compact interval.

Proof. By Lemma 1, $s_i \in S_i^{\boldsymbol{\chi}}$ if and only if some $k \in [0,1], g, b \in \Delta\Delta\Theta$ exist such that

(i′) $kb + (1-k)g \in \mathcal{P}(\mu_0)$,

(ii′) $(1-k)\int \mu \,\mathrm{d}g(\mu) \geqslant (\mathbf{1}-\boldsymbol{\chi})\mu_0$,

(iii′) $g\{V \ni s_i\} = b\{w \leq s_i\} = 1$.

Then, the existence of (k, g, b) satisfying (i′-iii′) immediately implies the existence of (k, γ, β) satisfying (i-iii) by setting $\gamma := \int \mu \,\mathrm{d}g(\mu), \beta := \int \mu \,\mathrm{d}b(\mu)$. Conversely, let (k, γ, β) satisfy (i-iii). By Lipnowski and Ravid's (2020) Theorem 2 and Corollary 3:

- Some $g \in \mathcal{P}(\gamma)$ exists with $g\{V \ni s_i\} = 1$ if and only if $s_i \in [\underline{w}(\gamma), \bar{v}(\gamma)]$,
- Some $b \in \mathcal{P}(\beta)$ exists with $b\{w \leq s_i\} = 1$ if and only if $s_i \geqslant \underline{w}(\beta)$.

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Thus, we obtain the desired characterization.

Finally, to show the "moreover" part, rewrite the above characterization of $S_i^{\boldsymbol{\chi}}$ as follows. Let $\mathcal{M}$ be the set of Borel measures on Θ and $\mathcal{G} := \{\eta \in \mathcal{M}\colon\ (\mathbf{1}-\boldsymbol{\chi})\mu_0 \leq \eta \leq \mu_0\}$, a compact convex subset. Define the functions

$$\tilde{v}\colon \mathcal{M} \to \mathbb{R} \qquad \eta \mapsto \begin{cases} \bar{v}\left(\frac{\eta}{\eta(\Theta)}\right) & : \eta \neq 0 \\ \max \bar{v}(\Delta\Theta) & : \eta = 0 \end{cases}$$

$$\tilde{w}\colon \mathcal{M} \to \mathbb{R} \qquad \eta \mapsto \begin{cases} \underline{w}\left(\frac{\eta}{\eta(\Theta)}\right) & : \eta \neq 0 \\ \min \underline{w}(\Delta\Theta) & : \eta = 0 \end{cases}$$

$$\kappa\colon \mathcal{G} \to \mathbb{R}$$
$$\eta \mapsto \tilde{v}(\eta) - \max\{\tilde{w}(\eta), \tilde{w}(\mu_0 - \eta)\}.$$

Then, the above characterization implies $s_i \in S_i^{\boldsymbol{\chi}}$ if and only if some $\eta \in \mathcal{G}$ exists such that $s_i \in [\max\{\tilde{w}(\eta), \tilde{w}(\mu_0 - \eta), \tilde{v}(\eta)]$, because $(k, \gamma, \beta) \mapsto (1-k)\gamma$ is a surjection from the subset of $(k, \gamma, \beta) \in [0,1] \times \Delta\Theta^2$ satisfying (i-ii) to $\mathcal{G}$. But this means $S_i^{\boldsymbol{\chi}} = \tau(\mathcal{G}^*)$, where $\mathcal{G}^* := \kappa^{-1}([0,\infty))$ and τ is a correspondence defined as

$$\tau\colon \mathcal{G}^* \rightrightarrows \mathbb{R}$$
$$\eta \mapsto [\max\{\tilde{w}(\eta), \tilde{w}(\mu_0 - \eta)\}, \tilde{v}(\eta)].$$

We now proceed to show $S_i^{\boldsymbol{\chi}}$ is a nonempty compact interval. First, observe that κ is upper semicontinuous and quasiconcave—because both $\bar{v}$ and $-\underline{w}$ are, and therefore so are $\tilde{v}$ and $-\tilde{w}$. Hence, the set $\kappa^{-1}([0,\infty)) = \mathcal{G}^*$ is compact and convex, and it is also nonempty because it contains μ_0. Second, note τ is a Kakutani correspondence because it is compact-convex-valued by definition, nonempty-valued by the definition of $\mathcal{G}^*$, and upper hemicontinuous by upper (resp. lower) semicontinuity of $\tilde{v}$ ($\tilde{w}$). Hence, the result follows from Fact 1. □

Building on the previous two lemmas, the following result shows the set of ex-ante $\boldsymbol{\chi}$-equilibrium payoffs for S is convex.

Lemma 7. *The set $\{\boldsymbol{\chi} s_o + (\mathbf{1}-\boldsymbol{\chi}) s_i\colon (p, s_o, s_i)$ is a $\boldsymbol{\chi}$-equilibrium summary$\}$ of ex-ante $\boldsymbol{\chi}$-equilibrium payoffs is a nonempty compact interval.*

Proof. Define the correspondence

$$\varsigma\colon S_i^{\boldsymbol{\chi}} \rightrightarrows \mathbb{R}$$
$$s_i \mapsto \{\boldsymbol{\chi}(\mu_0)s_o + [1-\boldsymbol{\chi}(\mu_0)]\, s_i\colon\ (p, s_o, s_i) \text{ is a } \chi\text{-equilibrium summary}\}\,.$$

We show ς is a Kakutani correspondence, which will give the desired result in light of Fact 1 and Lemma 6.

First, ς is nonempty-valued by the definition of $S_i^{\boldsymbol{\chi}}$. Second, the graph of ς is compact as a continuous image of the compact space X defined in the proof of Corollary 2. Therefore, ς is compact-valued and upper hemicontinuous.

Finally, we show ς is convex-valued. Fix any $s_i \in S_i^{\boldsymbol{\chi}}$, $s, s' \in \varsigma(s_i)$, $\lambda \in (0,1)$. By Lemma 1, some $k, k' \in [0,1], g, g', b, b' \in \Delta\Delta\Theta$ exist such that

$$kb + (1-k)g \in \mathcal{P}(\mu_0), \qquad k'b' + (1-k')g' \in \mathcal{P}(\mu_0),$$
$$(1-k)\int \mu \,\mathrm{d}g(\mu) \geqslant (\mathbf{1}-\boldsymbol{\chi})\mu_0, \qquad (1-k')\int \mu \,\mathrm{d}g'(\mu) \geqslant (\mathbf{1}-\boldsymbol{\chi})\mu_0,$$
$$s \in (1-k)s_i + k\int_{\mathrm{supp}(b)} s_i \wedge V \,\mathrm{d}b, \qquad s' \in (1-k')s_i + k'\int_{\mathrm{supp}(b')} s_i \wedge V \,\mathrm{d}b'.$$

Let $s^* \coloneqq \lambda s + (1-\lambda)s'$, $k^* \coloneqq \lambda k + (1-\lambda)k'$, $g^* \coloneqq \lambda\frac{1-k}{1-k^*}g + (1-\lambda)\frac{1-k'}{1-k^*}g'$, and $b^* \coloneqq \lambda\frac{k}{k^*}b + (1-\lambda)\frac{k'}{k^*}b'$. Then, by Lemma 1, (k^*, g^*, b^*) witness a $\boldsymbol{\chi}$-equilibrium with expected payoff s^* influencing payoff s_i. Thus, $\varsigma(s_i)$ is convex. □

C.2 Signaling Credibility

In this section, we present the formal analysis of the modified game in which S can signal her credibility through the choice of the official reporting protocol.

We start by introducing the modified game and notation. At the beginning, S privately learns her credibility type $t \in T = \{o, i\}$, that is, if the message will be determined according to the official protocol ($t = o$) or if it will be possible to influence it ($t = i$). Then, the game proceeds exactly as in our main model.

We focus on perfect Bayesian equilibria in which R's off-path beliefs satisfy a standard *"no signaling what you don't know"* restriction. To formalize the relevant solution concept, let Ξ denote the set of all official reporting protocols, that is, measurable maps $\xi\colon \Theta \to \Delta M$; endow Ξ with some measurable structure such that singletons are measurable. Then, let $(\xi_o, \xi_i) \in \Xi^T$ denote S's signaling strategy;[24] let the measurable maps $\boldsymbol{\sigma}\colon \Theta \times T \times \Xi \to \Delta M$, $\boldsymbol{\alpha}\colon M \times \Xi \to \Delta A$, and $\boldsymbol{\pi}\colon M \times \Xi \to \Delta\Theta$ denote S's influencing strategy, R's strategy, and R's belief map, respectively, that take into account the announced reporting protocol $\xi \in \Xi$; and let $\tilde{\boldsymbol{\chi}} : \Theta \times \Xi \to [0,1]$ denote R's measurable belief mapping from an announced official reporting protocol to S's posterior credibility. Then, a $\boldsymbol{\chi}$ **signaling PBE ($\boldsymbol{\chi}$-SPBE)** is a tuple $(\xi_o, \xi_i, \boldsymbol{\sigma}, \boldsymbol{\alpha}, \tilde{\boldsymbol{\chi}}, \boldsymbol{\pi})$ such that (letting $\boldsymbol{\sigma}_\xi := \boldsymbol{\sigma}(\cdot, \xi)$ and similarly for $\boldsymbol{\alpha}$, $\tilde{\boldsymbol{\chi}}$, and $\boldsymbol{\pi}$):

1. $\tilde{\boldsymbol{\chi}}$ is derived from $\boldsymbol{\chi}$ via Bayes' rule, given signal $t \mapsto \xi_t$, whenever possible.

2. $(\xi, \boldsymbol{\sigma}_\xi, \boldsymbol{\alpha}_\xi, \boldsymbol{\pi}_\xi)$ is a $\tilde{\boldsymbol{\chi}}_\xi$-equilibrium (for prior μ_0) for each $\xi \in \Xi$.

3. ξ_t maximizes $\boldsymbol{s}_t(\cdot)$ over Ξ, for each $t \in \{o, i\}$, where

$$\begin{aligned}\boldsymbol{s_o}\colon \Xi &\to \mathbb{R}\\ \xi &\mapsto \int_\Theta \int_M u_S(\boldsymbol{\alpha}_\xi(m))\,\mathrm{d}\xi(m|\cdot)\,\mathrm{d}\mu_0,\\ \boldsymbol{s_i}\colon \Xi &\to \mathbb{R}\\ \xi &\mapsto \int_\Theta \int_M u_S(\boldsymbol{\alpha}_\xi(m))\,\mathrm{d}\boldsymbol{\sigma}_\xi(m|\cdot)\,\mathrm{d}\mu_0.\end{aligned}$$

We call $(\max_\Xi \boldsymbol{s_o}, \max_\Xi \boldsymbol{s_i}) = (\boldsymbol{s_o}(\xi_o), \boldsymbol{s_i}(\xi_i))$ the corresponding S **payoff vector**. A **pooling $\boldsymbol{\chi}$-SPBE** is one in which $\xi_o = \xi_i$.

Note the above definition is equivalent to perfect Bayesian equilibria in which R updates joint beliefs over $T \times \Theta$, satisfying a "no signaling what you don't know" refinement. Indeed, because the official protocol announcement cannot convey information about the state, the T-marginal $\tilde{\boldsymbol{\chi}}_\xi$ (where we identify a belief on T with the probability it puts on o) determines the joint belief $\tilde{\boldsymbol{\chi}}_\xi \otimes \mu_0$. Then, given the form of R's incentive constraints after a message is received, it is enough to track only the Θ-marginal $\boldsymbol{\pi}_\xi$.

Recall, $\underline{w} : \Delta\Theta \to \mathbb{R}$ is the quasiconvex envelope of w, that is, the pointwise highest quasiconvex and lower semi-continuous function that is everywhere below w, or, equivalently, $-\underline{w} = \overline{-w}$. It follows directly from Lipnowski and Ravid (2020) that a sender-worst 0-equilibrium exists and delivers S payoff $\underline{w}(\mu_0)$.

[24]To simplify notation, here we focus on pure signaling strategies. Analogous results holds for mixed signaling strategies.

The following proposition establishes the equivalence between $\boldsymbol{\chi}$-equilibrium payoff vectors and $\boldsymbol{\chi}$-SPBE payoff vectors for S.

Proposition 4. *Fixing $(s_o, s_i) \in \mathbb{R}^2$, the following are equivalent:*

(a) *(s_o, s_i) is a $\boldsymbol{\chi}$-SPBE S payoff vector;*

(b) *(s_o, s_i) is a pooling $\boldsymbol{\chi}$-SPBE S payoff vector;*

(c) *(p, s_o, s_i) is a $\boldsymbol{\chi}$-equilibrium summary for some $p \in \mathcal{P}(\mu_0)$.*

Proof. First, (b) trivially implies (a).

Now, let us show (c) implies (b). To do so, consider some χ-equilibrium $(\xi, \sigma, \alpha, \pi)$ generating summary (p, s_o, s_i). Observe that for each $\xi' \in \Xi \setminus \{\xi\}$, some uncountable Borel $M_{\xi'} \subset M$ exists such that $\int_\Theta \xi'(M_{\xi'}|\cdot)\,\mathrm{d}\mu_0 = 0$.[25] It then follows readily from Theorem 2 of Lipnowski and Ravid (2020) that some 0-equilibrium $(\xi', \sigma_{\xi'}, \alpha_{\xi'}, \pi_{\xi'})$ exists giving S payoff $\underline{w}(\mu_0)$ with messages restricted to $M_{\xi'}$, that is, with $\boldsymbol{\sigma}_{\xi'}(M_{\xi'}|\cdot) = \mathbf{1}$. We now proceed to construct a pooling $\boldsymbol{\chi}$-SPBE. Define an influencing sender strategy $\boldsymbol{\sigma}$ and credibility belief function $\tilde{\boldsymbol{\chi}}$ by letting, for each $\xi' \in \Xi$,

$$(\boldsymbol{\sigma}_{\xi'},\ \tilde{\boldsymbol{\chi}}_{\xi'}) := \begin{cases} (\sigma,\ \boldsymbol{\chi}) & :\ \xi' = \xi \\ (\sigma_{\xi'},\ \mathbf{0}) & :\ \xi' \neq \xi. \end{cases}$$

Next, fix some $\mu_* \in \operatorname{argmin}_{\Delta\Theta} w$ and some R best response a_* to μ_* with $u_S(a_*) = w(\mu_*)$. Define a receiver strategy $\boldsymbol{\alpha}$ and belief map (concerning the state) $\boldsymbol{\pi}$ by letting, for each $\xi' \in \Xi$ and $m \in M$,

$$(\boldsymbol{\alpha}_{\xi'}(m),\ \boldsymbol{\pi}_{\xi'}(m)) := \begin{cases} (\alpha(m),\ \pi(m)) & :\ \xi' = \xi \\ (\alpha_{\xi'}(m),\ \pi_{\xi'}(m)) & :\ \xi' \neq \xi,\ m \notin M_{\xi'} \\ (\delta_{a_*},\ \mu_*) & :\ \xi' \neq \xi,\ m \in M_{\xi'}. \end{cases}$$

By construction, $(\xi, \xi, \boldsymbol{\sigma}, \boldsymbol{\alpha}, \tilde{\boldsymbol{\chi}}, \boldsymbol{\pi})$ satisfies conditions 1 and 2 of the definition of $\boldsymbol{\chi}$-SPBE. Moreover, observe that, by Lemma 6, some $\gamma, \beta \in \Delta\Theta$ exist such that $s_i \geq$

[25]For any Borel probability measure η on $[0,1]$, construct an uncountable Borel η-null $X \subseteq [0,1]$ as follows. First, express $\eta = \lambda\eta_d + (1-\lambda)\eta_c$ for some $\lambda \in [0,1]$ and $\eta_d, \eta_c \in \Delta[0,1]$ with η_d discrete and η_c atomless; define the co-countable set $\hat{X} := \{x \in [0,1] :\ \eta_d\{x\} = 0\}$. Let F denote the (continuous) CDF of η_c. If F is constant on some nondegenerate interval $I \subseteq [0,1]$, then $X := \hat{X} \cap I$ is as desired. Otherwise, $X := \hat{X} \cap F^{-1}(\mathcal{C})$ is as desired, where $\mathcal{C} \subset [0,1]$ is the Cantor set.

Finally, such $M_{\xi'}$ exists because $\int_\Theta \xi'\,\mathrm{d}\mu_0$ is a Borel probability measure on M, and the measurable space M is isomorphic to $[0,1]$ by the Borel isomorphism theorem.

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$\max\{\underline{w}(\beta), \underline{w}(\gamma)\}$ and $\mu_0 \in \text{co}\{\gamma, \beta\}$. Hence, $s_i \geq \underline{w}(\mu_0)$ because $\underline{w}$ is quasiconvex. Therefore, condition 3 of the definition of a $\boldsymbol{\chi}$-SPBE is satisfied because $\boldsymbol{s_i}(\xi) = s_i \geqslant \underline{w}(\mu_0) = \boldsymbol{s_i}(\xi')$ and $\boldsymbol{s_o}(\xi) = s_o \geqslant \min_{\Delta\Theta} w = \boldsymbol{s_o}(\xi')$ for all $\xi' \in \Xi \setminus \{\xi\}$. Therefore, $(\xi, \xi, \boldsymbol{\sigma}, \boldsymbol{\alpha}, \tilde{\boldsymbol{\chi}}, \boldsymbol{\pi})$ is a pooling $\boldsymbol{\chi}$-SPBE with S's payoff vector (s_o, s_i) as desired.

It remains to be shown (a) implies (c). To that end, suppose (s_o, s_i) is some $\boldsymbol{\chi}$-SPBE payoff vector, as witnessed by $\boldsymbol{\chi}$-SPBE $(\xi_o, \xi_i, \boldsymbol{\sigma}, \boldsymbol{\alpha}, \tilde{\boldsymbol{\chi}}, \boldsymbol{\pi})$ generating payoff vector (s_o, s_i), and let the functions $\boldsymbol{s_o}, \boldsymbol{s_i}$ be as defined in the definition of a $\boldsymbol{\chi}$-SPBE; recall $\boldsymbol{s_o}, \boldsymbol{s_i} \leq s_i$ and $\boldsymbol{s_i}(\xi_i) = s_i$. For any $\xi \in \Xi$ with $\tilde{\boldsymbol{\chi}}_\xi = 1$, that $\boldsymbol{s_i}(\xi) \leq s_i$ implies we can assume without loss (modifying $\boldsymbol{\alpha}_\xi(m)$ and $\boldsymbol{\pi}_\xi(m)$ for some $m \in M$ with $\int_\Theta \xi(m|\cdot)\, \mathrm{d}\mu_0 = 0$, and modifying $\boldsymbol{\sigma}_\xi$) that $\boldsymbol{s_i}(\xi) = s_i$. Therefore, $\boldsymbol{s_i}(\xi_i) = \boldsymbol{s_i}(\xi_o) = s_i$. Thus, for each $\xi \in \{\xi_o, \xi_i\}$, Lemma 1 delivers $k_\xi \in [0,1]$ and $g_\xi, b_\xi \in \Delta\Delta\Theta$ satisfying

$$
\begin{aligned}
&k_\xi b_\xi + (1-k_\xi) g_\xi \in \mathcal{P}(\mu_0), \\
&(1-k_\xi) \int \mu \, \mathrm{d}g_\xi(\mu) \geqslant (\mathbf{1} - \tilde{\boldsymbol{\chi}}_\xi)\mu_0, \\
&g_\xi\{s_i \in V\} = b_\xi\{s_i \geq \min V\} = 1, \\
&s_i - \boldsymbol{s_o}(\xi) \in \tfrac{k_\xi}{\tilde{\boldsymbol{\chi}}_\xi} \left[s_i - \int s_i \wedge V \, \mathrm{d}b_\xi \right].
\end{aligned}
$$

But then consider

$$
\begin{aligned}
k &:= \boldsymbol{\chi}(\mu_0) k_{\xi_o} + [1 - \boldsymbol{\chi}(\mu_0)]\, k_{\xi_i} \in [0,1), \\
b &:= \tfrac{\boldsymbol{\chi}(\mu_0) k_{\xi_o}}{k} b_{\xi_o} + \left[1 - \tfrac{\boldsymbol{\chi}(\mu_0) k_{\xi_o}}{k}\right] b_{\xi_i} \in \Delta\Delta\Theta, \\
g &:= \left(1 - \tfrac{[1-\boldsymbol{\chi}(\mu_0)](1-k_{\xi_i})}{1-k}\right) g_{\xi_o} + \tfrac{[1-\boldsymbol{\chi}(\mu_0)](1-k_{\xi_i})}{1-k} g_{\xi_i} \in \Delta\Delta\Theta.
\end{aligned}
$$

Direct computations with (k, g, b) then show, by Lemma 1, that $(kb + (1-k)g, s_o, s_i)$ is a $\boldsymbol{\chi}$-equilibrium summary. □

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